ALGEBRA for KIDS

• Adding and Subtracting —

I'm thinking of a number, and its name is "x". If $x+1=3$ then what is x? (That is, if we add 1 to x we get 3. What number is one less than three?) Write your answer like this: $\boxed{x=2}$ (That is, x is two!)

OK, now I'm thinking of another number, and its name is "y". If $y+2=4$, what is y? $\boxed{\quad y = \qquad\qquad}$

Now I'm thinking of a number whose name is "A". If $A-1=2$, what is A? (That is, if we *subtract* 1 from A we get 2. What number is one *more* than two?) $\boxed{\quad A = \qquad\qquad}$

I'll stop saying, "I'm thinking of a number, and its name is. . . " now. Any symbol I use like x or y or A or z or anything that is obviously not a word in a sentence will be meant as a "symbol" for an unknown number which you are supposed to figure out.

If $Z-3=2$, what is Z? $\boxed{\qquad\qquad\qquad}$

If $x+4=9$, what is x? $\boxed{\qquad\qquad\qquad}$

You can do lots of these by yourself. The "algebra rule" for doing this kind of problem is that you can **add** or **subtract** the same thing from both sides of an equation and the equation is still true. (An equation is one of these things with an "equal sign" [=] in the middle, like $x+4=9$.) Go back and check the problems you have done so far and notice that you can get the answer just by adding or subtracting the right number from both sides. For instance, in the first problem ($x+1=3$) we subtract 1 from both sides ($x+1-1=3-1$) and the $+1$ and -1 on the left side cancel each other out so that both disappear and we have $(x = 3-1)$. But $3 - 1 = 2$ on the right side, so we have $(x = 2)$ which is the answer! You probably can solve these easily "by inspection" without worring about the algebra rule, but it is interesting to know another way.

This works just as well for *other symbols* as it does for numbers: If $x + y = y - 1$, I can subtract y from both sides to get $\boxed{x = -1}$. Simple, eh?

Here's an example of how you would *use* algebra to solve a *practical problem*: if you have ten video games after just buying three, how many did you have before you bought the new ones? Let the answer be x. Then the equation that describes what happened is $x+3 = 10$, and the solution is $\boxed{x = 7}$.

• Multiplying and Dividing —

Now let's do some multiplying and dividing. Suppose $2x = 4$. What is x? (That is, two *times* x is four. What number do we multiply by 2 to get 4?) $\boxed{}$

If $3y = 9$, what is y? $\boxed{}$

If $5A = 20$, what is A? $\boxed{}$

The trick here (other than just figuring it out by thinking, "Hmm, what number do I have to multiply by 5 to get 20?") is the algebra rule that you can **divide** both sides of an equation by the same thing and the resulting equation is still true. So if I have $2x = 4$, I can divide both sides by 2 which cancels the 2 multiplying x on the left side (leaving just x) and turns the 4 into a 2 on the right side: $x = 2$, the answer!

This also works fine for symbols as well as numbers: Suppose $ax + b = b + a$. Then first I subtract b from both sides, to get $ax = a$. (We always try to use *single characters* for symbols, so that "ax" means a *times* x or the *product* of a and x.) From $ax = a$ we get the answer by dividing both sides by a, which cancels the a multiplying x on the left side and turns the a on the right side into a 1: $\boxed{x = 1}$ is the answer!

Note that anything divided by *itself* is 1, as in $a/a = 1$. (The way we write "B divided by C" is B/C, just as for numbers, like $4/2 = 2$.)

Try one for yourself: If $2XY + 1 = 2Y + 1$, what is X? (This is a little harder because X is multiplied by *both* the number 2 *and* the symbol Y, but you can do it the same way, either by dividing by 2 and then by Y or by dividing by $2Y$ all at once.) $\boxed{\qquad\qquad\qquad}$

Note that we *don't have to know what* Y *is*! It "drops out" of this equation! (The answer is $X = 1$ again. Did you get it right?) There are other cases where it *won't* drop out and you get an answer for X "*in terms* of Y". An example would be $xY = 1$, for which $x = 1/Y$ is the answer for x in terms of Y. (We could also "solve" the equation for Y in terms of x: $Y = 1/x$.)

The same thing works backwards, for equations like $X/2 = 3$. There the rule is that you can **multiply** both sides of an equation by the same thing and the resultant equation is still true. In this case we multiply both sides by 2. This cancels the $/2$ on the left side and turns the 3 on the right side into a 6. The answer is $\boxed{X = 6}$, which you could probably get just as easily by inspection ("Half of *what* is 3?"), but again it is nice to know the "rigorous" way of doing it, especially if you want to solve a more difficult equation like

$$\frac{(X-1)}{3} = 1$$

where the parentheses "$(\dots)$" indicate that what is inside them $(X-1)$ is *all* divided by 3.

Can you do this one? First multiply both sides by 3, then add 1 to both sides. What is your answer? $\boxed{}$

How about this? If $(aX + b)/2 = a$, what is X in terms of b? $\boxed{}$

Here's how this might be *applied* to another simple *practical* problem: Suppose your friend put some money in the bank at 10% *interest* [the " % " symbol is read "percent"]. This means that for every dollar he gives the bank to keep for one year, at the end of the year the bank gives him back the dollar *plus ten cents* "interest" on his dollar. At the end of the year, the bank gives him $220. *How much did he "deposit" [put in the bank] originally?* Let's let D be the number of dollars he deposited a year ago. For each dollar he put in, he gets back one dollar plus 1/10 of another dollar [ten cents is 1/10 of a dollar!]. But there are D dollars to start with, so we must have $D(1 + 1/10) = 220$. Now we can play a little trick with $1 + 1/10$: Since $1 = 10/10$ [any number divided by itself is 1], $1 + 1/10 = 10/10 + 1/10 = 11/10$ and so our equation reads $11D/10 = 220$. Now we multiply both sides by 10 to get $11D = 2200$ and then divide both sides by 11 to get $\boxed{D = 200}$, which is the answer. Your friend put $200 in the bank a year ago!

• Square Roots —

The *square* of a number means *the number multiplied by itself.* It is written with a little ^{2} just above and to the right, like this: $3^2 = 3 \times 3 = 9$. (Three threes are nine, right?) The reason it is called the *square* is because that is how we calculate the *area* [size] of a square, flat surface from the length of a side. The area A is equal to the *square* of the length ℓ of a side:

$$A = \ell \times \ell = \ell^2$$

You can see this easily by looking at a picture of four *square pizzas* of different sizes.[1]

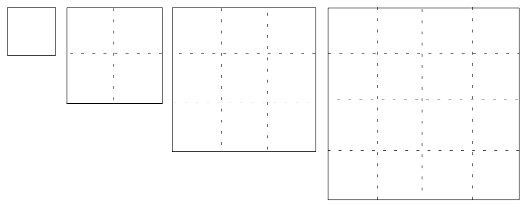

Figure 1: Four *square pizzas*. The first, on the left, is one inch on a side, making one small bite. The second is two inches on a side, making four small bites. How many bites in the third, which is three inches on a side? ☐ How about the last one, on the right, which is four inches on a side? ☐

Suppose we want to order one of these square pizzas big enough to feed five kids who each want 20 bites. How long should its sides be, if one square inch makes a bite? Well, if each side is ℓ inches long then the total area is ℓ^2 square inches, or ℓ^2 bites. We can make up an equation: $\ell^2 = 5 \times 20 = 100$. What number multiplied by itself is 100? The answer is $\boxed{\ell = 10}$ — the pizza should be ten inches on a side.

Suppose we have the equation $x^2 = 4$ and we want to know what x is. The equation tells us that $x \times x = 4$. What number multiplied by itself

[1]Round pizzas obey a similar rule, but it is easier to do the arithmetic for square ones. This is probably why the people who sell pizzas like to sell round ones, to make it harder to figure out how much you are really getting!

makes 4? Well, $2 \times 2 = 4$, so $x = 2$ is one answer. But there is another one! Suppose you multiply $(-2) \times (-2)$. the rules about multiplying *positive* numbers (like +2) and *negative* numbers (like −2) are as follows:

- A positive times a positive is a positive. $[+ \times + = +]$
- A positive times a negative is a negative. $[+ \times - = -]$
- A negative times a positive is a negative. $[- \times + = -]$
- A negative times a negative is a positive. $[- \times - = +]$

Thus $(-2) \times (-2) = (+4)$ [the square of (-2) is 4] so there are *two* correct solutions to the equation $x^2 = 4$, $x = 2$ and $x = -2$. A short way of writing this is to put the "+" and "−" signs together into a "±" ["plus-or-minus"] sign: $\boxed{x = \pm 2}$ which reads, "x is plus or minus two."

Now you try one: if $Y^2 = 9$, what is Y? $\boxed{Y = \quad}$

How about this: if $A^2 = 16$, what is A? $\boxed{\quad}$

Now let's put together all we have learned so far. If $x^2 - 1 = 15$, what is x? $\boxed{\quad}$

What if $2x^2 = 8$? [Note that $2x^2$ means $2 \times (x^2)$, *not* $(2 \times x)^2$.]

$\boxed{\quad}$

There is a special name for "the number which gives a when you multiply it by itself:" it is called the *square root* of a, and it has a special symbol too: $\sqrt{a}$. Thus $\sqrt{a} \times \sqrt{a} = a$ and of course $\sqrt{x^2} = x$. So what we are doing in these problems is using the algebra rule that says you can take the **square root** of both sides of an equation and the resultant equation is still true.

This will also work on a number whose square root is not an integer. [*Integers* are whole numbers like $1, 2, 3, 4, \ldots$, as opposed to fractions or "irrational" numbers, which we will see later.] For instance, $\sqrt{25} = 5$, but $\sqrt{2}$ is not an integer. This does not mean it is not a real number, but it is a very complicated one. In "decimal" notation it is $\sqrt{2} = 1.414213562373\ldots$, where the " $\ldots$ " represents *an infinite number of significant decimal places*. This is what we call an *irrational* number, but it is still a perfectly good number. If you run across it, just leave it in the form $\sqrt{2}$. That is as good an answer as any. Same for $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and so on.

Thus if $x^2 = 3$, then $\boxed{x = \pm\sqrt{3}}$. [Note that we still have to remember those $\pm$ signs!]

You try one: if $x^2 = b$, what is x in terms of b? $\boxed{}$

Now try this: if $a\,x^2 - b = 0$, what is x in terms of a and b? $\boxed{}$

• Quadratic Equations —

A *quadratic* equation "in x" means one with x^2 and maybe x in it, but no "higher powers" of x like $x \times x \times x = x^3$. For instance, $x^2 - 1 = 0$ is a quadratic equation in x but $3x - 1 = 8$ is *not*. A quadratic equation in x *must* contain x^2 (possibly multiplied or divided by some number) but it may or may not include x or some constant. Let's look at some of the different kinds of quadratic equations in x.

The first kind we have already seen. These are quadratic equations that have x^2 and some constant (like 3 or b) but no term in x. All the examples so far are of this form. A trivial but important case is $x^2 = 0$, for which the solution is $\boxed{x = 0}$. [Note that we don't have to say ± 0 because $+0$ and -0 are the same thing!]

The second kind has a term proportional to x, as in $3x^2 - 6x = 0$. First we can divide both sides by 3 to get $x^2 - 2x = 0$. [Zero multiplied or divided by anything is still zero!] This "simplifies" the equation a bit. Then we ask ourselves the question, "How can this equation be true?" [That is, *for what values of* x is this equation correct?] Well, there are two ways: first, x can be zero. Then both of the terms on the left side are zero, so they add up to zero and the equation reads $0 = 0$, which is certainly true. This is often called the "trivial solution," $\boxed{x = 0}$, because it is not very interesting; but it is a "true" solution anyway. To see the other solution, suppose we add $2x$ to both sides of the latest equation, to get $x^2 = 2x$. We can then *divide both sides by* x to get $\boxed{x = 2}$, which is the second solution. [$x^2/x = (x \times x)/x = x$.] Note that this time there is no $\pm$ sign because we are never taking a square root! But there are two answers, just as before.

We call the solutions to a quadratic equation the "roots" of the equation, to suggest the similarity to *square* roots. You try a few:

What are the roots of the equation $x^2 - x = 0$? $\boxed{x = \quad}$ and $\boxed{x = \quad}$

What are the roots of the equation $5y^2 = 25y$? $\boxed{y = \quad}$ and $\boxed{y = \quad}$

What are the roots of the equation $a\,x^2 - b\,x = 0$? $\boxed{\quad}$ and $\boxed{\quad}$

The most complicated type of quadratic equation in x is one that has a term proportional to x^2, another term proportional to x and a third *nonzero constant* term. An example would be $x^2 - 2x + 1 = 0$. [When they get complicated, we like to put all the nonzero terms on the left side of the equation and keep a zero on the right.] The trick to solving an equation like this is to see if it can be written as the product of two *pairs* of terms. In this case,

$$(x-1) \times (x-1) = x \times (x-1) - 1 \times (x-1) = x^2 - x - x + 1 = x^2 - 2x + 1$$

which means that our original equation is the same as $(x-1)(x-1) = 0$. [We can leave out the $\times$ ("times") symbol when we multiply together two things in parentheses: $(\ldots)(\ldots) = (\ldots) \times (\ldots)$.] So the equation will be true only if $(x-1) = 0$ or if $\boxed{x = 1}$. Note that $x = -1$ is *not* a right answer in this case!

A tougher example would be this: $x^2 - 3x + 2 = 0$. This has two answers; can you find them? $\boxed{x = \quad}$ and $\boxed{x = \quad}$

A *really* tough example would be $2y^2 - 5y + 2 = 0$. This also has two answers; can you find them? $\boxed{\quad}$ and $\boxed{\quad}$

Unlike the other kinds of algebra problems we have done before, these kind rely on "seeing the answer" or making a good guess. There is nothing wrong with this — seeing the answer is always the quickest and easiest way to solve a problem, as long as your guess is *right* ! Sometimes the best approach to a problem is to make a guess and then *check it* to see if it works; you can do this a few times before it gets to be a waste of time. But if after a while your guesses aren't working, you need a more rigourous approach to the problem. Fortunately, this kind of problem has a *general* solution, which is a little complicated, but if you memorize it you can always solve any quadratic equation just by "plugging in the parameters" that make it special. We will derive this below, but for now let's think about some more pieces to the puzzle.

• Imaginary Numbers —

OK, now you have solved problems where the answer was an *integer* (like 1 or 0 or -7) and problems where the answer was the *ratio* of two integers [the *ratio* of two things means one divided by the other] — which we call a "rational number" because it seems so reasonable — and you have even solved problems where the answer was an "irrational number" [one that *cannot* be expressed as the ratio of any two integers], like $\sqrt{2}$. All these different kinds of numbers are *real* numbers, even if they seem pretty unusual, because you can do arithmetic with them in the usual way. You have also seen that *symbols* like A or b can be used in answers just as if they were numbers, which lets you solve a whole bunch of different problems "formally" at the same time. Now it's time to tackle something really weird.

A long time ago, someone was telling her friends about this algebra stuff, showing that both $x = +1$ and $x = -1$ were solutions to the equation $x^2 = 1$, and someone said, "What about $x^2 = -1$? What would be the solution to that equation?" Well, all the other mathematicians laughed and said, "Boy, are you dumb! There *isn't any real number* that gives -1 when you multiply it by itself!" The first mathematician thought about it for a moment and said, "That is true, but wouldn't it be nice if there *were* such a number? You could solve *any* quadratic equation then!"

Her friends all laughed again and said, "Boy, do you have a great imagination!" And as they walked off laughing, she thought to herself, "OK, maybe I do; why not *call* the square root of -1 an *imaginary* number? I can even give it a special name "i" to signify that it is *i*maginary!" And she started figuring out how i would behave.

[This isn't really how i was invented, but it makes a good story!]

Anyway, if we have a solution to the equation $x^2 = -1$, then we can indeed solve *any* quadratic equation. For instance, because the square root of a product is the product of the square roots, like

$$\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9} = \pm 2 \times \pm 3 = \pm 6$$

we can *separate out* i like this:

$$\sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4} \times \sqrt{-1} = \pm 2\, i$$

where $2\,i$ means $2 \times i$ just as for any other symbol like $2x \equiv 2 \times x$. [The symbol " $\equiv$ " is used as shorthand for "...means the same thing as...."]

Let's try a few. If $x^2 = -9$, what is x? $\boxed{x = \pm \quad\quad}$

If $y^2 = -16$, what is y? $\boxed{y = \quad\quad}$

What if $z^2 - 1 = -37$? $\boxed{\quad\quad\quad}$

Great! Now (with a lot of work) we can actually write down the answer to *all possible quadratic equations* in *one formula* !

• The Quadratic Theorem —

The *most general possible quadratic equation* (after we have put all the terms on the lefthand side) looks like this:

$$\boxed{a\,x^2 \;+\; b\,x \;+\; c \;=\; 0}$$

where a, b and c are symbols representing some numbers that go into the equation. For instance, $4x^2 + 3x + 1 = 0$ would be a case where

$a = 4$, $b = 3$ and $c = 1$. For another example, $x^2 - 1 = 0$, would be a case with $a = 1$, $b = 0$ and $c = -1$. So if we can "solve" this equation for x *in terms of* a, b and c, then we will be able to quickly convert the result into the specific solution for whatever quadratic equation we want to solve, just by substituting a, b and c into the result! Let's try.

First let's divide through by a to get

$$x^2 + \frac{b}{a}\,x + \frac{c}{a} = 0$$

Next we subtract $\frac{c}{a}$ from both sides to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now we play an important trick known as "completing the square:" consider the equation $(x+d)^2 = x^2 + 2dx + d^2$. If we subtract d^2 from both sides we get $(x+d)^2 - d^2 = x^2 + 2dx$, which looks a lot like the left side of our previous equation, if only $2d$ were the same thing as $\frac{b}{a}$ — that is, if only $d = \frac{b}{2a}$. Well, let's put that in!

$$x^2 + \frac{b}{a}\,x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

But the right side of this equation must be equal to the right side of the old equation:

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a}$$

and if we add $\left(\frac{b}{2a}\right)^2$ to both sides we get

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}.$$

Now we can take the square root of both sides [remembering our $\pm$ sign] and get

$$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

— from both sides of which we subtract $\frac{b}{2a}$ to get

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}},$$

which is the answer, but still sort of messy. We can simplify a little by noting that

$$\frac{c}{a} = \frac{4ac}{4a^2} \qquad \text{and} \qquad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}, \qquad \text{so} \qquad \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

and therefore

$$\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a}$$

so that our answer now reads

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or just

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

which is known as THE QUADRATIC THEOREM.

This formula, messy as it may look, is worth remembering, because you can use it to solve *any* quadratic equation! First you put your equation into the form $ax^2 + bx + c = 0$ and figure out what the values of a, b and c are. then you plug those values of a, b and c into the QUADRATIC THEOREM and out pops the answer!

Let's do an example. Suppose we want to solve $2y^2 - 5y + 2 = 0$ for y. This is in the standard form with $a = 2$, $b = -5$ and $c = 2$. The answer is thus

$$y = \frac{+5 \pm \sqrt{25 - 4 \times 2 \times 2}}{2 \times 2} = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

so there are two answers, $\boxed{y = \frac{8}{4} = 2}$ and $\boxed{y = \frac{2}{4} = \frac{1}{2}}$.

You can go back over all the examples mentioned so far and show that each one is a case of the QUADRATIC THEOREM for some values of a, b and c.

Go ahead and think up lots of cases on your own! As long as you put them in the standard form $ax^2+bx+c=0$, you can always get the answer!