Solutions to Problem Set No. 1

UBC Metro Vancouver Physics Circle 2018

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Problem 1

At this moment of time, the electron experiences two forces: (1) the electrostatic force of attraction to the proton and (2) the centripetal force due to the external magnetic field. The sum of these forces must resemble circular motion; thus, we must compute our answer by taking into consideration angular acceleration.

$$\sum F = m_{\rm e}\vec{a} = \frac{m_{\rm e}v_{\rm e}^2}{R}$$
$$ev_{\rm e}B - \frac{ke^2}{R^2} = \frac{m_{\rm e}v_{\rm e}^2}{R}$$
$$ev_{\rm e}BR^2 - ke^2 = m_{\rm e}v_{\rm e}^2R$$

In the equation above, e is the elementary charge. By equating this equation to zero, we can use the quadratic formula to solve for the speed of the electron.

$$0 = (m_{\rm e}R) v_{\rm e}^2 - (eBR^2) v_{\rm e} + ke^2$$
$$v_{\rm e} = \frac{eBR^2 \pm \sqrt{e^2 B^2 R^4 - 4m_{\rm e}Rke^2}}{2m_{\rm e}R}$$

To achieve a real solution, and thus a stable circular motion,

$$e^{2}B^{2}R^{4} - 4m_{e}Rke^{2} \ge 0$$

$$e^{2}B^{2}R^{4} \ge 4m_{e}Rke^{2} \qquad R \ge \sqrt[3]{\frac{4(9.11 \times 10^{-31})(9 \times 10^{9})}{(0.1)^{2}}}$$

$$R \ge \sqrt[3]{\frac{4m_{e}k}{B^{2}}} \qquad R \ge 1.5 \times 10^{-6} \text{ m}$$

Therefore, the minimum R must be about 1.5 μ m.

Problem 2

To find the total amount of time, we must consider two factors: (1) the time it takes for the rock to hit the ocean and (2) the time it takes for the sound of the splash to travel back to the physicist. Let's refer to the diagram below.



First, we'll need to calculate the time it takes for the rock to hit the water. Let's call this time $t_{\rm air}$. If we consider downward the positive direction, then the expression for $t_{\rm air}$ is,

$$\mathbf{d} = \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2$$

$$h = -v_0 \sin \theta t + \frac{1}{2} g t^2$$

$$0 = \frac{1}{2} g t^2 - v_0 \sin \theta t - h$$

$$t = \frac{v_0 \sin \theta \pm \sqrt{v_0^2 \sin^2 \theta - 4\left(\frac{1}{2}g\right)\left(-h\right)}}{2\left(\frac{1}{2}g\right)}$$

$$t_{air} = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}$$
(Eqn. 1)

We will need to choose the positive solution since choosing the negative solution will result to a negative time. This, however, is only part of the answer. We must also calculate the time it takes for the sound to reach the physicist, t_s . As the rock hits the ocean, sound waves travel in all directions. But the one that reaches the physicist in the shortest amount of time would be the sound that travels in a straight line from the point of contact to the physicist. Since the speed of sound travels at constant speed in a given medium, we can calculate its time by dividing the distance the sound wave travels by its speed in air. Calculating this distance with respect to the original parameters is the tricky part. Let's refer to the diagram in the previous page.

In order to calculate the distance, we will need to use the Pythagorean Theorem, since we can form a right angle triangle. However, we first need to find the horizontal range of the rock, R, which is the horizontal displacement as it hits the ocean.

$$\mathbf{d} = \mathbf{v}t$$

$$R = (v_0 \cos \theta) t_{\text{air}}$$

$$R = \frac{v_0 \cos \theta \left(v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh} \right)}{g}$$

$$R = \frac{v_0^2 \sin \theta \cos \theta + v_0 \cos \theta \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}$$
(Eqn. 2)

Now that we have R, we can calculate the distance the sound wave travels in order to get to the physicist by using Equation 2.

$$\begin{split} d_s &= \sqrt{R^2 + h^2} \\ &= \sqrt{\left(\frac{v_0^2 \sin \theta \cos \theta + v_0 \cos \theta \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g}\right)^2 + h^2} \\ &= \sqrt{\frac{v_0^4 \sin^2 \theta \cos^2 \theta + 2v_0^3 \sin \theta \cos^2 \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + v_0^2 \cos^2 \theta \left(v_0^2 \sin^2 \theta + 2gh\right)}{g^2}} + h^2 \\ &= \sqrt{\frac{v_0^4 \sin^2 \theta \cos^2 \theta + 2v_0^3 \sin \theta \cos^2 \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + v_0^4 \sin^2 \theta \cos^2 \theta + 2ghv_0^2 \cos^2 \theta}{g^2}} + h^2 \\ &= \sqrt{\frac{2v_0^4 \sin^2 \theta \cos^2 \theta + 2v_0^3 \sin \theta \cos^2 \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + 2ghv_0^2 \cos^2 \theta}{g^2}} + h^2 \\ &= \sqrt{\frac{2v_0^2 \cos^2 \theta \left(v_0^2 \sin^2 \theta + v_0 \sin \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + gh\right)}{g^2}} + h^2 \\ &= \sqrt{\frac{2v_0^2 \cos^2 \theta \left(v_0^2 \sin^2 \theta + v_0 \sin \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + gh\right)}{g^2}} \\ &= \frac{\sqrt{2v_0^2 \cos^2 \theta \left(v_0^2 \sin^2 \theta + v_0 \sin \theta \sqrt{v_0^2 \sin^2 \theta + 2gh} + gh\right)} + g^2h^2}}{g} \end{split}$$

We can now calculate the time it takes for the sound wave to reach the physicist.

$$t_{s} = \frac{d_{s}}{v_{s}} = \frac{\sqrt{2v_{0}^{2}\cos^{2}\theta \left(v_{0}^{2}\sin^{2}\theta + v_{0}\sin\theta \sqrt{v_{0}^{2}\sin^{2}\theta + 2gh} + gh\right) + g^{2}h^{2}}}{gv_{s}}$$
(Eqn. 3)

By combining Equation 1 and Equation 3, we get the expression for the total time.

 $t_{\text{total}} = t_{\text{air}} + t_s$

$$=\frac{v_0\sin\theta+\sqrt{v_0^2\sin^2\theta+2gh}}{g}+\frac{\sqrt{2v_0^2\cos^2\theta\left(v_0^2\sin^2\theta+v_0\sin\theta\sqrt{v_0^2\sin^2\theta+2gh}+gh\right)+g^2h^2}}{gv_s}$$
$$=k_1\left(\beta+\sqrt{\beta^2+2\gamma}\right)+k_2\left(\sqrt{2\alpha^2\left(\beta^2+\beta\sqrt{\beta^2+2\gamma}+\gamma\right)+\gamma^2}\right)$$

Where the following functions and constants are,

$$\alpha(v_0, \theta) = v_0 \cos \theta \qquad \qquad k_1 = \frac{1}{g} = \frac{1}{9.8} = 0.102 \text{ s}^2 \text{ m}^{-1}$$
$$\beta(v_0, \theta) = v_0 \sin \theta \qquad \qquad k_2 = \frac{1}{gv_s} = \frac{1}{9.8 \times 343} = 3 \times 10^{-4} \text{ s}^3 \text{ m}^{-2}$$
$$\gamma(h) = gh$$

Problem 3

To find the x and y coordinates of the particle as it hits the detection screen, we must do this problem one step at a time, starting with the particle going through the electric field. Since we are assuming this is a uniform electric field, the particle will experience a constant electric force,

$\mathbf{F} = q \cdot \mathbf{E}$

Since the charge is positive, the net force on this particle will be the electric force applied downwards as the particle is within \vec{E} . Therefore, if we assume downwards is positive, the

acceleration in the y-direction will be,

$$a = \frac{qE}{m}$$

Note that acceleration in the x-direction is zero, so horizontal velocity will be constant. Let's now calculate how much vertical deflection, y_{def} , we have throughout \vec{E} . Throughout all calculations, let's assume downwards is the positive direction.

$$\mathbf{d} = \mathbf{v}_{\mathbf{0}}t + \frac{1}{2}\mathbf{a}t^{2}$$
$$y_{\text{def},E} = 0 + \frac{qE}{2m}t^{2}$$

Notice that $\mathbf{v}_0 t$ is zero since there is no initial vertical velocity. The only unknown is time, which we can calculate from our horizontal equation with no acceleration involved.

$$\mathbf{d} = \mathbf{v} \cdot t$$
$$l_E = v t_E$$
$$t_E = \frac{l_E}{v}$$

As a result, we will end up with,

$$y_{\text{def},E} = \frac{qE \, l_E^2}{2mv^2} \tag{Eqn. 1}$$

As the particle leaves the electric field and is now in the d_1 zone, it will experience no net force, and thus its velocity will be constant. The horizontal component of the velocity is still v, but the vertical component of velocity can be found. Let's go back into the l_E zone, and start at the beginning to calculate final speed.

$$\mathbf{v} = \mathbf{v_0} + \mathbf{a}t$$
$$v_{f,y} = 0 + at_E$$
$$v_{f,y} = \frac{qE\,l_E}{mv}$$

To calculate the vertical deflection in d_1 zone, we'll need to calculate the time from the horizontal equation just like above,

$$\mathbf{d_x} = \mathbf{v} \cdot t \qquad \qquad \mathbf{d_y} = \mathbf{v} \cdot t$$

$$d_1 = vt_1 \qquad \qquad y_{\text{def},d_1} = v_{f,y}t_1$$

$$t_1 = \frac{d_1}{v} \qquad \qquad y_{\text{def},d_1} = \frac{qEd_1l_E}{mv^2} \qquad (\text{Eqn. 2})$$

Now, the particle is going to approach the magnetic field. Before we do any calculations, we have to understand that charged particles follow a circular path when they enter a magnetic field. The reason for their circular motion is a result of the centripetal net force equivalent to the magnetic force,

$$\mathbf{F} = m \cdot \mathbf{a}$$
$$q \, \mathbf{v} \times \mathbf{B} = \frac{mv^2}{r}$$

When \mathbf{v} and \mathbf{B} are orthogonal to each other, the circle's radius is equivalent to

$$qvB = \frac{mv^2}{r}$$
$$r = \frac{mv}{qB}$$

However, when **v** and **B** are separated by an angle θ that is less than 90°, we get a helical motion parallel to **B**.



The velocity component that is parallel to **B** which is equivalent to $v \cos \theta$ will remain constant since there is no acceleration in that direction. The velocity component that is perpendicular to **B** which is equivalent to $v \sin \theta$, however, will be affected by the magnetic force and result in the particle's circular motion. As a result, the overall motion is helical.

The distance a particle moves in the direction parallel to \mathbf{B} is known as "pitch", p. Since we determined that motion parallel to \mathbf{B} is constant, we can calculate p by,

$$\mathbf{d} = \mathbf{v} \cdot t$$
$$p = v \cos(\theta) T$$

where T is the period of revolution for the particle. In order to calculate the particle's vertical deflection while it is in the magnetic field, we must use the notion of pitch in our computation.

Before we do any calculation, recall that up to this point, the particle's velocity components are

$$v_x = v$$
 $v_y = \frac{qE \, l_E}{mv}$

Consider the diagram below, representing the particle's path as it enters the magnetic field, extended for a length of l_B ; the dashed curve is the extrapolated path.



The point C is the centre of curvature and r is the radius of curvature. Since the particle does reach the detection screen, it is important to note that $r > l_B$. If this was not the

case, the particle will not be able to go past the magnetic field. We are also safe to assume that the particle will travel the whole length l_B since the question states the widths of the magnets are much greater than their lengths.

By using the Pythagorean theorem, we get

$$x' = \sqrt{r^2 - l_B^2}$$

and the angle to be

$$\alpha = \arctan \frac{l_B}{\sqrt{r^2 - l_B^2}}$$

In order to find the time the particle spends in the magnetic field before it exits, we can take a ratio of the arc to 2π , a full circle, and multiply it by the period of revolution, T.

$$d_{\rm arc} = r \alpha \qquad t_B = \frac{d_{\rm arc}}{2\pi} T$$
$$= r \arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right) \qquad = \frac{\arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right)}{2\pi} T$$

In circular motion, T can also be written as the following where v_c resembles the speed of constant circular motion,

$$T = \frac{2\pi r}{v_c}$$

Recall that this circular motion is essentially caused by the velocity component that is perpendicular to the magnetic field, $v \sin \theta$ in our previous diagram. Since this circular speed is equivalent to the horizontal velocity component, it is also equivalent to v which is the particle's original velocity as it was shot into the electric field in the beginning since $v_x = v$. As a result, the radius of curvature of this circle will be equivalent to

$$r = \frac{mv}{qB}$$

and thus, we can conclude that

$$t_B = \frac{\arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right)}{2\pi} T$$
$$= \frac{\arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right)}{2\pi} \left(\frac{2\pi r}{v}\right)$$
$$= \frac{\arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right)}{2\pi} \left(\frac{2\pi mv}{qBv}\right)$$
$$= \frac{m}{qB} \arctan\left(\frac{l_B}{\sqrt{r^2 - l_B^2}}\right)$$

With some mathematical manipulations and the fact that $r = \frac{mv}{qB}$, we get

$$t_B = \frac{m}{qB} \arctan\left(\frac{1}{\sqrt{\left(\frac{mv}{qB \, l_B}\right)^2 - 1}}\right)$$

Now that we have calculated the time the particle spends in the magnetic field before it exits, we are able to express the vertical deflection of the particle within the magnetic field.

$$y_{\text{def},B} = v_y t_B$$

$$y_{\text{def},B} = \left(\frac{qE \, l_E}{mv}\right) \cdot \frac{m}{qB} \arctan\left(\frac{1}{\sqrt{\left(\frac{mv}{qB \, l_B}\right)^2 - 1}}\right)$$

$$y_{\text{def},B} = \frac{E \, l_E}{Bv} \arctan\left(\frac{1}{\sqrt{\left(\frac{mv}{qB \, l_B}\right)^2 - 1}}\right)$$
(Eqn. 3)

Note that this is derived from the formula used to calculate the pitch.

The last part is to compute the deflection in the d_2 zone. Since the net force in this zone is zero, the velocity will be constant. Let's look at the particle's final path before it hits the detection screen.



Since the circular speed is v, it will remain v as it shoots out of the magnetic field. In fact, the angle remains α by using geometry. Since the net force here is zero, we can easily calculate how much time it takes for the particle to reach the detection screen.

$$\mathbf{d} = \mathbf{v} \cdot t$$
$$d_r = vt_2$$
$$\frac{d_2}{\cos \alpha} = vt_2$$
$$t_2 = \frac{d_2}{v \cos \alpha}$$

We can simplify $\cos \alpha$ to be

$$\cos \alpha = \frac{x'}{r}$$
$$= \frac{\sqrt{r^2 - l_B^2}}{r}$$
$$= \sqrt{1 - \left(\frac{l_B}{r}\right)^2} = \sqrt{1 - \left(\frac{qB \, l_B}{mv}\right)^2}$$

As a result, we can now get an expression for t_2 with our variables included

$$t_2 = \frac{d_2}{v\sqrt{1 - \left(\frac{qB\,l_B}{mv}\right)^2}}$$

To calculate how much y-deflection we have in d_2 zone, remember that our $v_y = \frac{qE l_E}{mv}$ even up to this point. Thus,

$$\mathbf{d_y} = \mathbf{v_y} \cdot t$$

$$y_{\text{def},d_2} = \left(\frac{qE \, l_E}{mv}\right) \cdot \left(\frac{d_2}{v\sqrt{1 - \left(\frac{qB \, l_B}{mv}\right)^2}}\right)$$

$$y_{\text{def},d_2} = \frac{qE \, l_E \, d_2}{mv^2 \sqrt{1 - \left(\frac{qB \, l_B}{mv}\right)^2}}$$
(Eqn. 4)

If we put Equations 1-4 all together, we get the expression for total vertical deflection downwards.

$$y_{\rm def} = \frac{qE \, l_E^2}{2mv^2} + \frac{qE d_1 l_E}{mv^2} + \frac{E \, l_E}{Bv} \arctan\left(\frac{1}{\sqrt{\left(\frac{mv}{qB \, l_B}\right)^2 - 1}}\right) + \frac{qE \, l_E \, d_2}{mv^2 \sqrt{1 - \left(\frac{qB \, l_B}{mv}\right)^2}}$$

The next task is to calculate the deflection of the particle into the screen. Therefore, the total x-deflection on the detection screen can be described by

$$\sum x_{\rm def} = x_{\rm def} + x'_{\rm def}$$

Where x_{def} and x'_{def} are portrayed in the previous figure.

$$\begin{aligned} x_{def} &= r - x' & x'_{def} &= d_2 \tan \alpha \\ &= r - \sqrt{r^2 - l_B^2} & = d_2 \cdot \frac{l_B}{\sqrt{r^2 - l_B^2}} \\ &= r \left[1 - \sqrt{1 - \left(\frac{l_B}{r}\right)^2} \right] & = d_2 \cdot \frac{1}{\sqrt{\left(\frac{r}{l_B}\right)^2 - 1}} \\ &= \frac{mv}{qB} \left[1 - \sqrt{1 - \left(\frac{qB \ l_B}{mv}\right)^2} \right] & = \frac{d_2}{\sqrt{\left(\frac{mv}{qB \ l_B}\right)^2 - 1}} \end{aligned}$$

Therefore, the total x-deflection will be,

$$\sum x_{\text{def}} = \frac{mv}{qB} \left[1 - \sqrt{1 - \left(\frac{qB \ l_B}{mv}\right)^2} \right] + \frac{d_2}{\sqrt{\left(\frac{mv}{qB \ l_B}\right)^2 - 1}}$$

As a result, the theoretically expected x- and y-coordinates on the detection screen will be:

$$x - \text{coordinate} = -\frac{mv}{qB} \left[1 - \sqrt{1 - \left(\frac{qB \ l_B}{mv}\right)^2} \right] - \frac{d_2}{\sqrt{\left(\frac{mv}{qB \ l_B}\right)^2 - 1}}$$
$$y - \text{coordinate} = -\frac{qE \ l_E^2}{2mv^2} - \frac{qE \ l_1 \ l_E}{mv^2} - \frac{E \ l_E}{Bv} \arctan\left(\frac{1}{\sqrt{\left(\frac{mv}{qB \ l_B}\right)^2 - 1}}\right) - \frac{qE \ l_E \ d_2}{mv^2\sqrt{1 - \left(\frac{qB \ l_B}{mv}\right)^2}}$$

The coordinates will both be negative since the particle was deflected downwards and into the page when referring to the original diagram. In the solution above, we have chosen to use the inverse tangent function to find the vertical deflection within the magnetic field. However, there are various ways to represent this answer, such as using the inverse sine or cosine function. As a result, the theoretical y-coordinate can also be represented as

$$y - \text{coordinate} = -\frac{qE \, l_E^2}{2mv^2} - \frac{qE d_1 l_E}{mv^2} - \frac{E \, l_E}{Bv} \operatorname{arcsin}\left(\frac{qB \, l_B}{mv}\right) - \frac{qE \, l_E \, d_2}{mv^2 \sqrt{1 - \left(\frac{qB \, l_B}{mv}\right)^2}}$$

or

$$y - \text{coordinate} = -\frac{qE\,l_E^2}{2mv^2} - \frac{qEd_1l_E}{mv^2} - \frac{E\,l_E}{Bv}\arccos\left(\sqrt{1 - \left(\frac{qB\,l_B}{mv}\right)^2}\right) - \frac{qE\,l_E\,d_2}{mv^2\sqrt{1 - \left(\frac{qB\,l_B}{mv}\right)^2}}$$