Solutions to Problem Set No. 2

UBC Metro Vancouver Physics Circle 2018

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Problem 1

(a) As explained in the problem, a charged particle while in motion and under the influence of a magnetic field will follow a circular path. The centripetal force is equivalent to the magnetic force.

$$\vec{F}_B = m \cdot \vec{a}$$

$$q(\vec{v} \times \vec{B}) = \frac{mv^2}{R}$$

Since $\vec{v}$ is perpendicular to $\vec{B}$, we can have:

$$q\,|\vec{v}|\,|\vec{B}| = \frac{mv^2}{R} \qquad\qquad T = \frac{2\pi R}{|\vec{v}|}$$

$$R = \frac{m\,|\vec{v}|}{q\,|\vec{B}|} \qquad\qquad T = \frac{2\pi}{|\vec{v}|} \cdot \frac{m\,|\vec{v}|}{q\,|\vec{B}|}$$

$$T = \frac{2\pi m}{q\,|\vec{B}|}$$

Therefore, the period of rotation is only dependent on the particle's charge and mass, and the magnetic field strength. This suggests that as the particle spirals out, its speed and radius both increase proportionally such that the period is constant.

(b) If the proton spirals out to reach the very edge of the cyclotron before it exits, that means it reached its maximum speed. Recall from the work in part (a) that

$$v = \frac{qR\,|\vec{B}|}{m}$$

$$\therefore\; v_{\max} = \frac{qR_{\max}\,|\vec{B}|}{m}$$

We can use this to find the maximum kinetic energy.

$$\begin{aligned} E_{\max} &= \frac{1}{2}m(v_{\max})^2 \\ &= \frac{1}{2}m \cdot \left(\frac{qR_{\max}\,|\vec{B}|}{m}\right)^2 \end{aligned}$$

When we manipulate this equation to solve for $R_{\max}$, we get

$$R_{\max} = \frac{\sqrt{2mE_{\max}}}{q\,|\vec{B}|}$$

In this equation, $R_{\max}$ will correspond to the radius of each dee. If we want to find their diameters, we must multiply this quantity by 2. So, for a proton that leaves the cyclotron with maximum energy of 25 MeV, the diameter of each dee must be:

$$\begin{aligned} D_{\max} &= 2\frac{\sqrt{2 \cdot 1.6726 \times 10^{-27}\,\mathrm{kg} \cdot 4.0054 \times 10^{-12}\,\mathrm{J}}}{1.6022 \times 10^{-19}\,\mathrm{C} \cdot 1.0\,\mathrm{T}} \\ &= 1.44\,\mathrm{m} \qquad\qquad (1\,\mathrm{eV} = 1.6022 \times 10^{-19}\,\mathrm{J}) \end{aligned}$$

As for the second part of this question, we can assume that the proton has negligibly small kinetic energy at the beginning. The kinetic energy, E_k, that the particle with charge q gains during the acceleration by the voltage ΔV is equal to:

$$E_k = q\Delta V$$

The proton has to go through the dees n-times to reach the required energy E_{max}

$$E_{\text{max}} = nE_k$$

Thus, the n in this case will equal to

$$n = \frac{E_{\text{max}}}{E_k} = \frac{E_{\text{max}}}{q\Delta V} = \frac{4.0054 \times 10^{-12}}{1.6022 \times 10^{-19} \cdot 50 \times 10^3} \approx 500$$

Thus, the proton has to loop about 250 times before it exits. Since $n = 500$, that's how many times the particle gets an energy boost. For each loop, it gets boosted twice.

Problem 2

To derive the relation mentioned in the question, we must first understand a key concept in quantum mechanics. Although matter can behave as a particle, it can also simultaneously behave as a wave. This concept was proposed by Louis de Broglie in 1924. He hypothesized that a moving particle has a characteristic wavelength, λ, dependent on its momentum, p, and vice versa. In fact, the constant relating these two quantities is Planck's constant, h. The de Broglie wavelength can be found by:

$$\lambda = \frac{h}{p}$$

Now, let's derive the relation using the principle of conservation of momentum. In this 2-dimensional problem, we need to deal with horizontal component of momentum and vertical component of momentum. Let's define the following:

$$p = \frac{h}{\lambda} = \text{momentum of incident photon.}$$
$$p_1 = \frac{h}{\lambda'} = \text{momentum of scattered photon at angle } \theta.$$
$$p_2 = \text{momentum of recoil electron at an angle } \phi.$$

The x component:

$$p = p_1 \cos\theta + p_2 \cos\phi$$

$$p_2 \cos\phi = p - p_1 \cos\theta$$

The y component:

$$p_1 \sin\theta = p_2 \sin\phi$$

Let's take a ratio:

$$\frac{p_2 \sin\phi}{p_2 \cos\phi} = \frac{p_1 \sin\theta}{p - p_1 \cos\theta}$$

$$\tan\phi = \frac{p_1 \sin\theta}{\frac{h}{\lambda} - p_1 \cos\theta}$$

$$= \frac{p_1 \sin\theta}{p_1 \sin\theta \left(\frac{h}{\lambda p_1 \sin\theta} - \cot\theta \right)}$$

$$= \frac{1}{\frac{h}{\lambda \frac{h}{\lambda'} \sin\theta} - \cot\theta}$$

$$= \frac{1}{\frac{\lambda'}{\lambda \sin\theta} - \cot\theta}$$

$$= \frac{\lambda \sin\theta}{\lambda' - \lambda \cos\theta}$$

$$= \frac{\lambda \sin\theta}{\Delta\lambda + \lambda - \lambda \cos\theta}$$

$$= \frac{\lambda \sin\theta}{\Delta\lambda + \lambda (1 - \cos\theta)}$$

$$= \frac{\lambda \sin\theta}{\lambda (1 - \cos\theta) + \frac{h}{m_e c} (1 - \cos\theta)}$$

$$= \frac{\sin\theta}{(1 - \cos\theta) + \frac{h}{\lambda m_e c} (1 - \cos\theta)}$$

$$= \frac{\sin\theta}{(1-\cos\theta) + \frac{h}{\frac{c}{f} m_e c}(1-\cos\theta)}$$

$$= \frac{\sin\theta}{(1-\cos\theta) + \frac{hf}{m_e c^2}(1-\cos\theta)}$$

$$= \frac{\sin\theta}{(1-\cos\theta)\left(1 + \frac{hf}{m_e c^2}\right)}$$

$$\left(1 + \frac{hf}{m_e c^2}\right)\tan\phi = \frac{\sin\theta}{1-\cos\theta}$$

By using the double-angle identities of sine and cosine,

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad\qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

we can derive the following

$$\left(1 + \frac{hf}{m_e c^2}\right)\tan\phi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

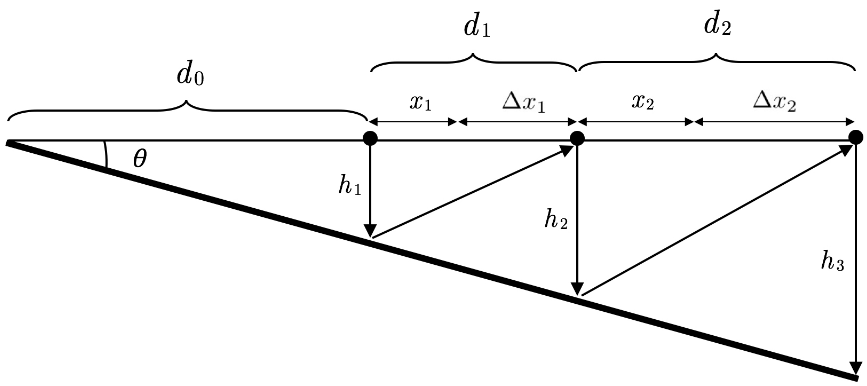
$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}$$

$$\therefore \quad \cot\frac{\theta}{2} = \left(1 + \frac{hf}{m_e c^2}\right)\tan\phi$$

Problem 3

(a) To answer this question, we must draw a detailed yet simple diagram that will help us. Let's represent the boat by a simple circle.



In the diagram above, let:

x_n = The distance the boat will travel when the n^{th} signal reaches the ocean floor.

$\hookrightarrow$ Let the time for this distance correspond to t_n.

Δx_n = The distance the boat will travel when the n^{th} signal returns from the ocean floor.

$\hookrightarrow$ Let the time for this distance correspond to $t_{\Delta n}$.

h_n = The vertical height travelled by the n^{th} signal.

d_0 = The horizontal distance travelled by the boat before it sends its first signal.

Now that we have defined a few things, we can begin. Our ultimate goal is to prove that

$$\frac{d_{n+1}}{d_n} = K$$

This stated in another way is

$$\frac{x_{n+1} + \Delta x_{n+1}}{x_n + \Delta x_n} = K$$

In this entire problem, it is very important that we assume $v_S \gg v_B$, as stated by the question. If these speeds were similar, then the signal will not travel straight down; it will travel at an angle that will significantly affect the results. It is also important to notice that the question states all speeds are constant. As a result, we are essentially only dealing with

$d = vt$. Let's start with x_1 and Δx_1.

$$x_1 = v_B t_1 \qquad\qquad h_1 = v_S t_1$$

$$\frac{x_1}{v_B} = \frac{h_1}{v_S}$$

$$x_1 = \frac{h_1 v_B}{v_S}$$

Let's call $d_{\text{return},1}$ to be the distance the first signal travels as it is coming back to the boat. Adjacent to this, we can also use the boat's distance travelled in that time.

$$\sqrt{h_1^2 + (x_1 + \Delta x_1)^2} = d_{\text{return},1} \qquad\qquad \Delta x_1 = v_B t_{\Delta 1}$$

$$\sqrt{h_1^2 + (x_1 + \Delta x_1)^2} = v_S t_{\Delta 1} \qquad\qquad \frac{\Delta x_1}{v_B} = t_{\Delta 1}$$

$$\sqrt{h_1^2 + (x_1 + \Delta x_1)^2} = \frac{v_S}{v_B}\Delta x_1$$

$$h_1^2 + x_1^2 + 2x_1\Delta x_1 + (\Delta x_1)^2 = \left(\frac{v_S}{v_B}\right)^2 (\Delta x_1)^2$$

If we collect all terms on one side, we get a quadratic equation. We can use the quadratic formula to solve for Δx_1:

$$0 = \left(\left(\frac{v_S}{v_B}\right)^2 - 1\right)(\Delta x_1)^2 - (2x_1)\Delta x_1 - \left(h_1^2 + x_1^2\right)$$

$$\Delta x_1 = \frac{2x_1 \pm \sqrt{4x_1^2 + 4\left(\left(\frac{v_S}{v_B}\right)^2 - 1\right)(h_1^2 + x_1^2)}}{2\left(\left(\frac{v_S}{v_B}\right)^2 - 1\right)}$$

$$\Delta x_1 = \frac{x_1 \pm \sqrt{x_1^2 + \left(\left(\frac{v_S}{v_B}\right)^2 - 1\right)(h_1^2 + x_1^2)}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

Here, Δx_1 will only have a valid solution if we pick the positive root.

$$\Delta x_1 = \frac{x_1 + \sqrt{x_1^2 + \left(h_1 \frac{v_S}{v_B}\right)^2 - h_1^2 + \left(x_1 \cdot \frac{v_S}{v_B}\right)^2 - x_1^2}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

$$\Delta x_1 = \frac{x_1 + \sqrt{\left(h_1 \frac{v_S}{v_B}\right)^2 + \left(x_1 \cdot \frac{v_S}{v_B}\right)^2 - h_1^2}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

Now, we can replace $x_1 = \frac{h_1 v_B}{v_S}$

$$\Delta x_1 = \frac{\frac{h_1 v_B}{v_S} + \sqrt{\left(h_1 \frac{v_S}{v_B}\right)^2 + \left(\frac{h_1 v_B}{v_S} \cdot \frac{v_S}{v_B}\right)^2 - h_1^2}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

$$\Delta x_1 = \frac{\frac{h_1 v_B}{v_S} + \sqrt{\left(h_1 \frac{v_S}{v_B}\right)^2 + h_1^2 - h_1^2}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

$$\Delta x_1 = \frac{\frac{h_1 v_B}{v_S} + \frac{h_1 v_S}{v_B}}{\left(\frac{v_S}{v_B}\right)^2 - 1}$$

$$\Delta x_1 = h_1 \cdot \left[\frac{\frac{v_S^2 + v_B^2}{v_S v_B}}{\frac{v_S^2 - v_B^2}{v_B^2}}\right]$$

$$\Delta x_1 = h_1 \cdot \left[\frac{v_S^2 + v_B^2}{v_S v_B} \cdot \frac{v_B^2}{v_S^2 - v_B^2}\right]$$

$$\Delta x_1 = h_1 \frac{v_B}{v_S} \left[\frac{v_S^2 + v_B^2}{v_S^2 - v_B^2} \right]$$

Let's call $\alpha = \dfrac{v_S^2 + v_B^2}{v_S^2 - v_B^2}$, and thus $\Delta x_1 = h_1 \alpha \dfrac{v_B}{v_S}$.

In fact, if we generalize this for any n^{th} signal, we get $x_n = h_n \dfrac{v_B}{v_S}$ and $\Delta x_n = h_n \alpha \dfrac{v_B}{v_S}$.

$$\therefore \quad x_n + \Delta x_n = h_n \frac{v_B}{v_S} (1 + \alpha) \tag{Eqn. 1}$$

Now, we must understand the behaviour of h_n with respect to the parameters that we're given.

$$h_1 = d_0 \tan\theta$$

$$\begin{aligned} h_2 &= (d_0 + x_1 + \Delta x_1) \tan\theta \\ &= \left(d_0 + h_1 \frac{v_B}{v_S}(1 + \alpha) \right) \tan\theta \end{aligned}$$

$$\begin{aligned} h_3 &= (d_0 + x_1 + \Delta x_1 + x_2 + \Delta x_2) \tan\theta \\ &= \left(d_0 + h_1 \frac{v_B}{v_S}(1 + \alpha) + h_2 \frac{v_B}{v_S}(1 + \alpha) \right) \tan\theta \\ &= \left(d_0 + \frac{v_B}{v_S}(1 + \alpha) \Big[h_1 + h_2 \Big] \right) \tan\theta \\ &\vdots \\ h_n &= \left(d_0 + \frac{v_B}{v_S}(1 + \alpha) \Big[h_1 + h_2 + \cdots + h_{n-2} + h_{n-1} \Big] \right) \tan\theta \\ &= \left(d_0 + \frac{v_B}{v_S}(1 + \alpha) \sum_{i=1}^{n-1} h_i \right) \tan\theta \end{aligned} \tag{Eqn. 2}$$

Thus, putting Equations 1 and 2 together, we get:

$$
\begin{aligned}
\frac{x_{n+1} + \Delta x_{n+1}}{x_n + \Delta x_n} &= \frac{h_{n+1} \dfrac{v_B}{v_S}\,(1+\alpha)}{h_n \dfrac{v_B}{v_S}\,(1+\alpha)} \\
&= \frac{h_{n+1}}{h_n} \\
&= \frac{\left(d_0 + \dfrac{v_B}{v_S}(1+\alpha)\displaystyle\sum_{i=1}^{n} h_i\right)\tan\theta}{\left(d_0 + \dfrac{v_B}{v_S}(1+\alpha)\displaystyle\sum_{i=1}^{n-1} h_i\right)\tan\theta} \\
&= \frac{d_0 + \dfrac{v_B}{v_S}(1+\alpha)\displaystyle\sum_{i=1}^{n-1} h_i + \dfrac{v_B}{v_S}(1+\alpha)h_n}{d_0 + \dfrac{v_B}{v_S}(1+\alpha)\displaystyle\sum_{i=1}^{n-1} h_i} \\
&= 1 + \frac{\dfrac{v_B}{v_S}(1+\alpha)h_n}{d_0 + \dfrac{v_B}{v_S}(1+\alpha)\displaystyle\sum_{i=1}^{n-1} h_i} \\
&= 1 + \frac{\dfrac{v_B}{v_S}(1+\alpha)h_n}{\dfrac{h_n}{\tan\theta}} \\
&= 1 + \frac{v_B}{v_S}\tan\theta\,(1+\alpha) \\
&= 1 + \frac{v_B}{v_S}\tan\theta\left(1 + \frac{v_S^2 + v_B^2}{v_S^2 - v_B^2}\right)
\end{aligned}
$$

$$= 1 + \tan\theta \, \frac{v_B}{v_S} \left(\frac{v_S^2 - v_B^2 + v_S^2 + v_B^2}{v_S^2 - v_B^2} \right)$$

$$= 1 + \frac{2 v_B v_S}{v_S^2 - v_B^2} \tan\theta$$

$$\therefore \; \frac{x_{n+1} + \Delta x_{n+1}}{x_n + \Delta x_n} = K = 1 + \frac{2 v_B v_S}{v_S^2 - v_B^2} \tan\theta$$

(b) In this second part, we must find the total distance, D_n, travelled by the boat for the n^{th} signal sent and received, since the first one was sent.

$$D_1 = d_1$$

$$\begin{aligned} D_2 &= d_1 + d_2 \\ &= d_1 \left(1 + \frac{d_2}{d_1} \right) \\ &= d_1(1 + K) = d_1(K + 1) \end{aligned}$$

$$\begin{aligned} D_3 &= d_1 + d_2 + d_3 \\ &= d_1 + d_2 \left(1 + \frac{d_3}{d_2} \right) \\ &= d_1 + d_2(1 + K) \\ &= d_1 \left(1 + \frac{d_2}{d_1}(1 + K) \right) \\ &= d_1 \Big(1 + K(1 + K) \Big) = d_1(K^2 + K + 1) \end{aligned}$$

$$\begin{aligned} D_4 &= d_1 + d_2 + d_3 + d_4 \\ &= d_1 + d_2 + d_3 \left(1 + \frac{d_4}{d_3} \right) \\ &= d_1 + d_2 + d_3(1 + K) \end{aligned}$$

$$= d_1 + d_2\left(1 + \frac{d_3}{d_2}(1+K)\right)$$

$$= d_1 + d_2\Big(1 + K(1+K)\Big)$$

$$= d_1\left(1 + \frac{d_2}{d_1}(K^2+K+1)\right)$$

$$= d_1\Big(1 + K(K^2+K+1)\Big) = d_1(K^3+K^2+K+1)$$

$$\vdots$$

$$D_n = d_1\sum_{i=0}^{n-1} K^i$$

$$= \Big(x_1 + \Delta x_1\Big)\sum_{i=0}^{n-1}\left(1 + \frac{2v_B v_S}{v_S^2 - v_B^2}\tan\theta\right)^i$$

$$= h_1\frac{2v_B v_S}{v_S^2 - v_B^2}\sum_{i=0}^{n-1}\left(1 + \frac{2v_B v_S}{v_S^2 - v_B^2}\tan\theta\right)^i \qquad \text{(Using Equation 1 here.)}$$

$$= d_0\tan\theta\,\frac{2v_B v_S}{v_S^2 - v_B^2}\sum_{i=0}^{n-1}\left(1 + \frac{2v_B v_S}{v_S^2 - v_B^2}\tan\theta\right)^i$$

$$\therefore\; D_n = \frac{2d_0 v_B v_S}{v_S^2 - v_B^2}\tan\theta\sum_{i=0}^{n-1}\left(1 + \frac{2v_B v_S}{v_S^2 - v_B^2}\tan\theta\right)^i$$

Ultimately, this becomes a special type of geometric series. The value of K resembles the common ratio, r. The total distance the boat travels since it sends its first signal always increases by a factor of K, and then add 1, all in terms of d_1. By definition since $K > 1$, the geometric series that describes the total distance will diverge; as $n \to \infty$, $D \to \infty$.