Problem Set No. 3

UBC Metro Vancouver Physics Circle 2018

April 5, 2018

Problem 1 — Hawking Radiation

We are asked to derive the Hawking-Bekenstein luminosity relation of a black hole using only Planck's constant (h), speed of light (c), Newton's gravitational constant (G), and mass of the black hole (M).

$$P = h^{\alpha} G^{\beta} M^{\gamma} c^{\delta}$$

Luminosity/Power has an SI unit of watt, W, but is described as energy per unit of time, J/s. If we establish every factor in terms of their SI base units, we get:

$$\begin{aligned}
[P] &= \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} = \left[ML^2T^{-3}\right] \\
[h] &= \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1} = \left[ML^2T^{-1}\right] \\
[G] &= \mathrm{kg}^{-1} \cdot \mathrm{m}^3 \cdot \mathrm{s}^{-2} = \left[M^{-1}L^3T^{-2}\right] \\
[M] &= \mathrm{kg} = [M] \\
[c] &= \mathrm{m} \cdot \mathrm{s}^{-1} = \left[LT^{-1}\right]
\end{aligned}$$

where $[M]$ is unit of mass, $[L]$ is unit of length, and $[T]$ is unit of time. Therefore, we achieve:

$$\left[ML^2T^{-3}\right] = \left[ML^2T^{-1}\right]^{\alpha} \left[M^{-1}L^3T^{-2}\right]^{\beta} [M]^{\gamma} \left[LT^{-1}\right]^{\delta}$$

However, we are told that luminosity is inversely proportional to the square of a black hole's mass. This essentially means that $\gamma = -2$.

$$\left[ML^2T^{-3}\right] = \left[ML^2T^{-1}\right]^{\alpha} \left[M^{-1}L^3T^{-2}\right]^{\beta} \left[M\right]^{-2} \left[LT^{-1}\right]^{\delta}$$

$$\begin{aligned}
\text{in } [M]\text{:} \quad & 1 = \alpha - \beta - 2 \\
\text{in } [L]\text{:} \quad & 2 = 2\alpha + 3\beta + \delta \\
\text{in } [T]\text{:} \quad & -3 = -\alpha - 2\beta - \delta
\end{aligned}$$

If we add the $[L]$ equation to the $[T]$ equation, we eliminate for δ and achieve $-1 = \alpha + \beta$. If we then add this equation to the $[M]$ equation, we eliminate β and get $\alpha = 1$. Solving for α means $\beta = -2$ and $\delta = 6$. Therefore, the following is true,

$$P = K\frac{hc^6}{G^2M^2}$$

where K is the proportionality constant.

Let's answer the second part of this question: What would the luminosity be for a black hole of 1 solar mass?

$$\begin{aligned}
P &= K\frac{hc^6}{G^2M_\odot^2} \\
&= K\frac{(6.626 \times 10^{-34})(3 \times 10^8)^6}{(6.67 \times 10^{-11})^2(2 \times 10^{30})^2} \\
&\approx 3 \times 10^{-23} \text{ W} && \text{(assuming } K = 1\text{)}
\end{aligned}$$

This is an incredibly small number! Since K is mentioned to be a small proportionality constant, this luminosity is only going to get smaller. With the mass of actual black holes being thousand to billion solar masses, this number is going to become significantly smaller. This is why it is extremely difficult to observe Hawking Radiation, and effectively a very good approximation of calling black holes as *black* holes!

Problem 2 — Muon Decay

(a) Let L_0 be the thickness of the atmosphere in an observer's rest frame (this is what we observe as ~ 10 km), and let L be the thickness of the atmosphere in the muon's moving frame. These two lengths are related to one another by

$$L = \frac{L_0}{\gamma}$$

where γ is the Lorentz factor, $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

If v is the muon's speed and τ is its lifetime, we can calculate how far the muon will reach in its own moving frame by $L = v\tau$. Therefore,

$$\begin{aligned}
L &= L_0\sqrt{1 - \frac{v^2}{c^2}} \\
v\tau &= L_0\sqrt{1 - \frac{v^2}{c^2}} \\
\left(\frac{v\tau}{L_0}\right)^2 &= 1 - \frac{v^2}{c^2} \\
v^2\left(\frac{\tau^2}{L_0^2} + \frac{1}{c^2}\right) &= 1 \\
v &= \frac{cL_0}{\sqrt{L_0^2 + c^2\tau^2}} \\
v &= \frac{c}{\sqrt{1 + \left(\frac{c\tau}{L_0}\right)^2}} \\
v &= \frac{c}{\sqrt{1 + \left(\frac{(3 \times 10^8)(2.2 \times 10^{-6})}{10^4}\right)^2}} \\
v &= 0.9978\, c
\end{aligned}$$

(b) In order to calculate the minimum energy needed for a muon to reach sea-level, we must use:

$$E = \gamma mc^2$$

We can use our answer in part (a) to calculate the Lorentz factor, and thus the minimum energy required:

$$\begin{aligned} E &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ &= \frac{\left(\frac{105.7 \text{ MeV}}{c^2}\right) \cdot c^2}{\sqrt{1 - (0.9978)^2}} \\ &\approx 1600 \text{ MeV} = 2.6 \times 10^{-10} \text{ J} \end{aligned}$$

Problem 3 — Stacked Blocks

Since no slipping occurs, the system will accelerate as a single unit according to Newton's 2^{nd} Law:

$$F = M_T a$$

$$a = \frac{F}{M_T} \qquad (M_T = \text{total mass})$$

We can find the coefficient of friction for the surface under the n^{th} block by noting that the n^{th} block and all blocks above it accelerate as a single unit as well with mass M_{above}, and that this acceleration is caused by the friction force resulting from the surface beneath the n^{th} block. Therefore, we can write

$$F_{fr} = M_{above} a = M_{above} \left(\frac{F}{M_T}\right)$$

We also know that the friction force is given by

$$F_{fr} = \mu_n^{n+1} M_{above} g$$

So we can write

$$M_{above}\left(\frac{F}{M_T}\right) = \mu_n^{n+1} M_{above} g$$

Solving for μ_n^{n+1} gives

$$\mu_n^{n+1} = \frac{M_{above} F}{M_{above} g M_T} = \frac{F}{M_T g}$$

Since the expression for μ_n^{n+1} does not depend on n, it must be the same for all levels, so option B is correct.

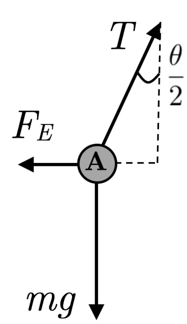
Problem 4 — Two Charged Spheres

First, note that at equilibrium there is no net torque on the system, which implies that the centre of mass of the two spheres must lie directly under the fixed point. In other words, both spheres are at the same height and have equal and opposite horizontal displacements from the centre of the system

$$\Delta x = L \sin\left(\frac{\theta}{2}\right)$$

The distance between the spheres is, therefore, given by $r = 2\Delta x = 2L\sin\left(\frac{\theta}{2}\right)$.

Let's draw a free-body diagram for sphere A:



where T is force of tension, F_E is the electrostatic force, and mg is just the force of gravity. Since the system is in equilibrium, the net force must be zero.

$$\sum F_x = 0 \qquad \sum F_y = 0$$

$$T \sin\left(\frac{\theta}{2}\right) = F_E \qquad T \cos\left(\frac{\theta}{2}\right) = mg$$

$$\frac{F_E}{mg} = \tan\left(\frac{\theta}{2}\right)$$

$$\left(\frac{kq_1q_2}{r^2}\right)\left(\frac{1}{mg}\right) = \tan\left(\frac{\theta}{2}\right)$$

$$\left(\frac{k(-Q)(-3Q)}{4L^2\sin^2\left(\frac{\theta}{2}\right)}\right)\left(\frac{1}{mg}\right) = \tan\left(\frac{\theta}{2}\right)$$

$$\frac{3kQ^2}{4mgL^2} = \tan\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right)$$

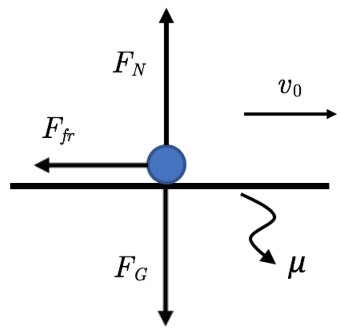
Using the small angle approximations, $\tan\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\theta}{2}\right) \approx \frac{1}{8}\theta^3$, we obtain:

$$\frac{1}{8}\theta^3 = \frac{3kQ^2}{4mgL^2}$$

$$\theta = \sqrt[3]{\frac{6kQ^2}{mgL^2}}$$

Problem 5 — Probability of Scoring a Goal

(a) First, let's calculate how much distance the ball will travel linearly before it stops. The only deceleration is due to the frictional force. Let's refer to the free body diagram below right when the string snaps.



As a result, we can set up our equation using Newton's Second Law of Motion, considering the deceleration direction positive, and find the distance the ball travels before it stops:

$$\sum \vec{F} = m\,\vec{a}$$

$$F_{fr} = ma \qquad\qquad v^2 = v_0^2 + 2\left(\vec{a}\cdot\vec{d}\right)$$

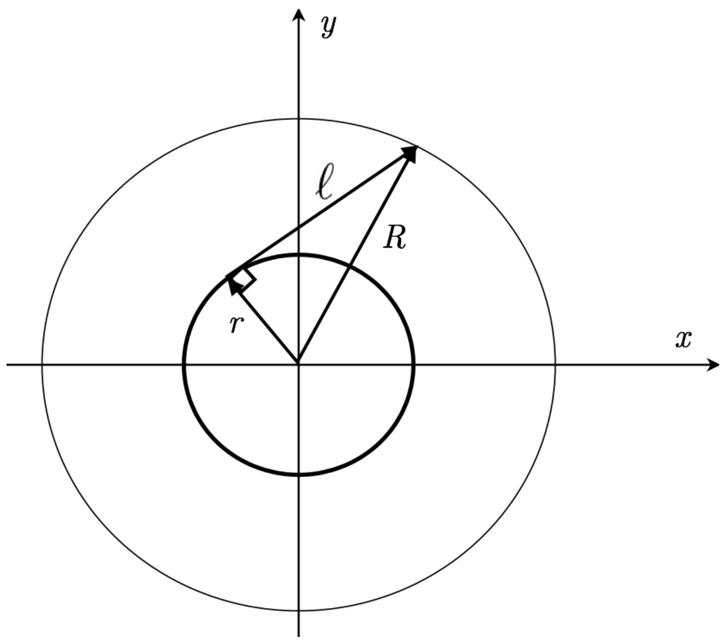
$$\mu F_N = ma \qquad\qquad 0 = \left(\frac{2\pi r}{T}\right)^2 + 2\mu g\,\vec{d}$$

$$\mu(mg) = ma \qquad\qquad |\vec{d}| = \frac{1}{2\mu g}\left(\frac{2\pi r}{T}\right)^2$$

$$\mu g = a \qquad\qquad |\vec{d}| = \frac{2\pi^2 r^2}{\mu g T^2}$$

As a result, let's call the distance the ball travels before it stops $\ell = \dfrac{2\pi^2 r^2}{\mu g T^2}$.

If we now take a random point where the string snaps, we can see that ℓ forms a right-angle triangle with r, and the hypotenuse of the triangle will be R. As a result, we can use the Pythagorean theorem to solve for R.



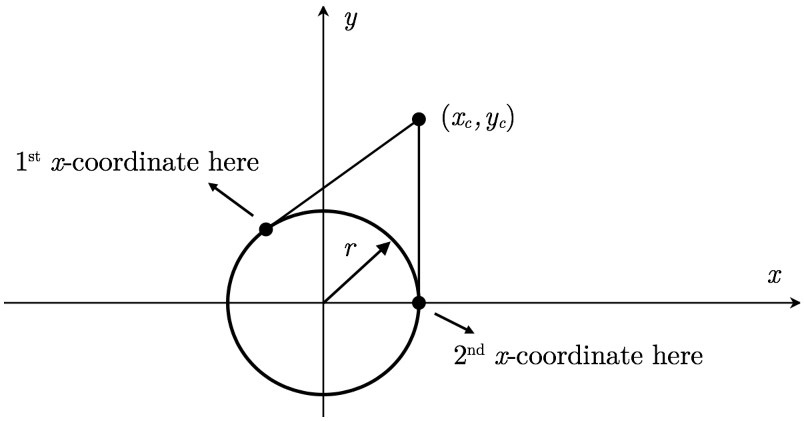
$$R^2 = \ell^2 + r^2$$

$$R = \sqrt{\ell^2 + r^2}$$

$$R = \sqrt{\left(\frac{2\pi^2 r^2}{\mu g T^2}\right)^2 + r^2}$$

$$R = r\sqrt{\frac{4\pi^4 r^2}{\mu^2 g^2 T^4} + 1}$$

(b) Essentially, we want to find out the x-coordinates on the circle such that their tangent lines pass through the point (x_c, y_c):



To find these x-coordinates, we must use the concept of perpendicular slopes. Namely, if a line has a slope m, then a second line that is perpendicular to the first must have a slope equivalent to $-\frac{1}{m}$; in other words, the slopes are negative reciprocals. In our circle, there are essentially two lines at play — (1) the line of length r (the radius) that connects the centre of the circle (at the origin) to our x-coordinate of interest, and (2) the tangent line originating at our x-coordinate of interest and going through the point (x_c, y_c). Since, by definition, these two lines are perpendicular to one another, their slopes are negative reciprocals of each other. For the first line (the radial line), the slope can always be described as $\frac{y}{x}$ no matter where our x-coordinate of interest lies; this is just rise over run. As a result, the slope of tangent line must be $m_\perp = -\frac{x}{y}$. Now, let's write an equation for the tangent line in the slope-point form.

$$
\begin{aligned}
y - y_c &= m_\perp (x - x_c) \\
y - y_c &= -\frac{x}{y}(x - x_c) \\
y^2 - y y_c &= -x^2 + x x_c \\
r^2 - x^2 - y y_c &= -x^2 + x x_c && \text{(Using } x^2 + y^2 = r^2\text{)} \\
r^2 - x x_c &= y y_c \\
r^2 - x x_c &= \pm\, y_c \sqrt{r^2 - x^2} \\
\left(r^2 - x x_c\right)^2 &= (y_c)^2 \left(r^2 - x^2\right)
\end{aligned}
$$

$$r^4 - 2r^2 x x_c + x^2 x_c^2 = r^2 y_c^2 - x^2 y_c^2$$

$$\left(x_c^2 + y_c^2\right) x^2 - \left(2r^2 x_c\right) x + \left(r^4 - r^2 y_c^2\right) = 0$$

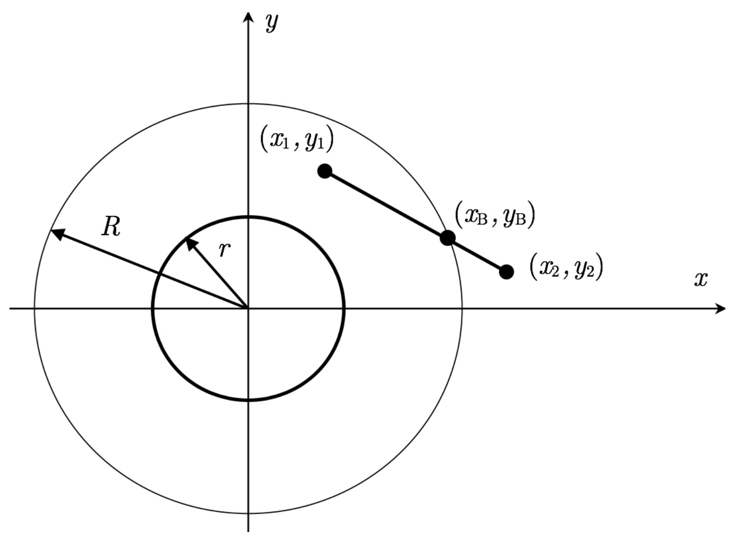
This looks like a quadratic equation, which we can use the quadratic formula for:

$$\begin{aligned}
x &= \frac{2r^2 x_c \pm \sqrt{4r^4 x_c^2 - 4\left(x_c^2 + y_c^2\right)\left(r^4 - r^2 y_c^2\right)}}{2\left(x_c^2 + y_c^2\right)} \\
&= \frac{2r^2 x_c \pm \sqrt{4r^4 x_c^2 - 4\left(r^4 x_c^2 - r^2 x_c^2 y_c^2 + r^4 y_c^2 - r^2 y_c^4\right)}}{2\left(x_c^2 + y_c^2\right)} \\
&= \frac{2r^2 x_c \pm \sqrt{4r^4 x_c^2 - 4r^4 x_c^2 + 4r^2 x_c^2 y_c^2 - 4r^4 y_c^2 + 4r^2 y_c^4}}{2\left(x_c^2 + y_c^2\right)} \\
&= \frac{2r^2 x_c \pm 2r y_c \sqrt{x_c^2 + y_c^2 - r^2}}{2\left(x_c^2 + y_c^2\right)} \\
&= \frac{r^2 x_c \pm r y_c \sqrt{x_c^2 + y_c^2 - r^2}}{x_c^2 + y_c^2}
\end{aligned}$$

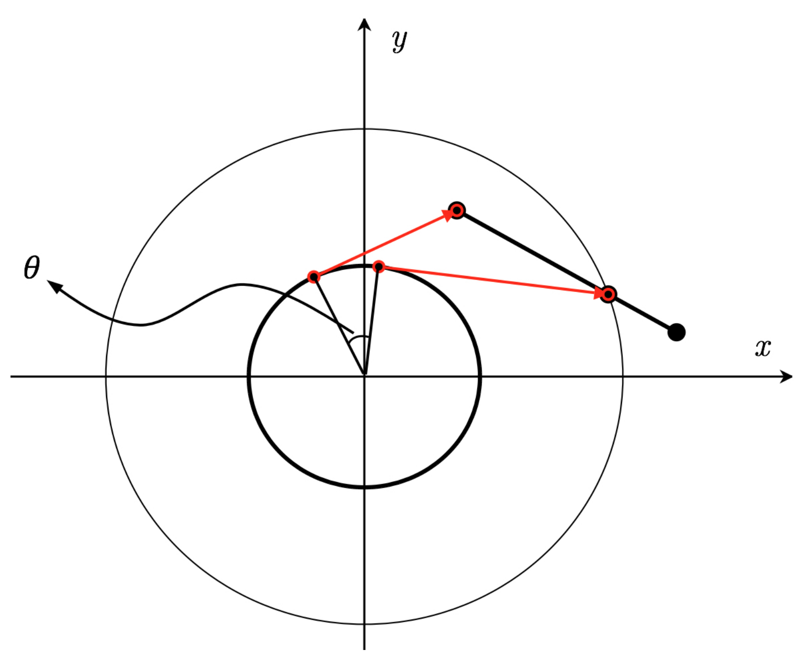
As a result, the x-coordinates on the circle $x^2 + y^2 = r^2$ that make tangents to exterior point (x_c, y_c) can be described by the formula

$$x = \frac{r^2 x_c \pm r y_c \sqrt{x_c^2 + y_c^2 - r^2}}{x_c^2 + y_c^2}$$

(c) To answer this question, we must utilize our answers from parts (a) and (b). The questions asks us about the probability of scoring a goal, where scoring a goal is defined as making contact with either the goal line or a goal post. When we look at the rules stated by the question, we notice that P_2 is always situated in the region $x^2 + y^2 > R$, meaning that it is outside the bound circle. This means that no matter where the ball enters its tangential trajectory, it will never reach P_2. Thus, what the ball will make contact with is restricted within the bound circle and, obviously, outside the original circle. Let's take two points in mind — the first will be the goal post described by P_1, while the second will be a point on the goal line that is situated right on the bound circle; let's call this point (x_B, y_B).



To calculate the probability of scoring a goal, we need to consider points P_1 and (x_B, y_B) as our extremes, find the tangents on the original circle associated with these extremes, and then compute the probability by dividing the angle associated with that arc by 2π. If we refer to the diagram below, then the probability would be $P = \dfrac{\theta}{2\pi}$.



Before we do any calculation, we have to find out how to describe the point (x_B, y_B), which is associated with the goal and the bound circle. Let's use the slope-point form to describe an equation for the goal line.

$$y - y_1 = m(x - x_1) \qquad \text{where} \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

Now, we must find the intersection between the linear function above and the bound circle, described by $x^2 + y^2 = R^2$.

$$\begin{aligned}
y &= m(x - x_1) + y_1 \\
y^2 &= \Big(m(x - x_1) + y_1\Big)^2 \\
y^2 &= m^2(x - x_1)^2 + 2my_1(x - x_1) + y_1^2 \\
R^2 - x^2 &= m^2(x - x_1)^2 + 2my_1(x - x_1) + y_1^2 && \text{(Using } x^2 + y^2 = R^2\text{)} \\
R^2 - x^2 &= m^2x^2 - 2m^2xx_1 + m^2x_1^2 + 2mxy_1 - 2mx_1y_1 + y_1^2 \\
0 &= (m^2 + 1)x^2 + (2my_1 - 2m^2x_1)x + (y_1^2 - 2mx_1y_1 + m^2x_1^2 - R^2)
\end{aligned}$$

Again, we see another quadratic equation which we can use the quadratic formula for:

$$x = \frac{2m^2x_1 - 2my_1 \pm \sqrt{(2my_1 - 2m^2x_1)^2 - 4(m^2 + 1)(y_1^2 - 2mx_1y_1 + m^2x_1^2 - R^2)}}{2(m^2 + 1)}$$

Since this is a long expression, let's simplify the square-root by its components:

$$\begin{aligned}
\text{Comp}_1 &= (2my_1 - 2m^2x_1)^2 \\
&= 4m^2y_1^2 - 8m^3x_1y_1 + 4m^4x_1^2
\end{aligned}$$

$$\begin{aligned}
\text{Comp}_2 &= -4(m^2 + 1)(y_1^2 - 2mx_1y_1 + m^2x_1^2 - R^2) \\
&= -4(m^2y_1^2 - 2m^3x_1y_1 + m^4x_1^2 - m^2R^2 + y_1^2 - 2mx_1y_1 + m^2x_1^2 - R^2)
\end{aligned}$$

$$= -4m^2y_1^2 + 8m^3x_1y_1 - 4m^4x_1^2 + 4m^2R^2 - 4y_1^2 + 8mx_1y_1 - 4m^2x_1^2 + 4R^2$$

$$\begin{aligned}\text{Comp}_1 + \text{Comp}_2 &= 4m^2R^2 - 4y_1^2 + 8mx_1y_1 - 4m^2x_1^2 + 4R^2 \\ &= 4R^2(m^2+1) - 4(mx_1 - y_1)^2\end{aligned}$$

Therefore, putting it all together, factoring the 4's out of the square-root and cancelling all the 2's, yields:

$$x = \frac{m(mx_1 - y_1) + \sqrt{R^2(m^2+1) - (mx_1 - y_1)^2}}{m^2+1}$$

This above equation describes the x-coordinate of the point (x_B, y_B). Note: We have to take the positive root since we are looking for a positive solution.

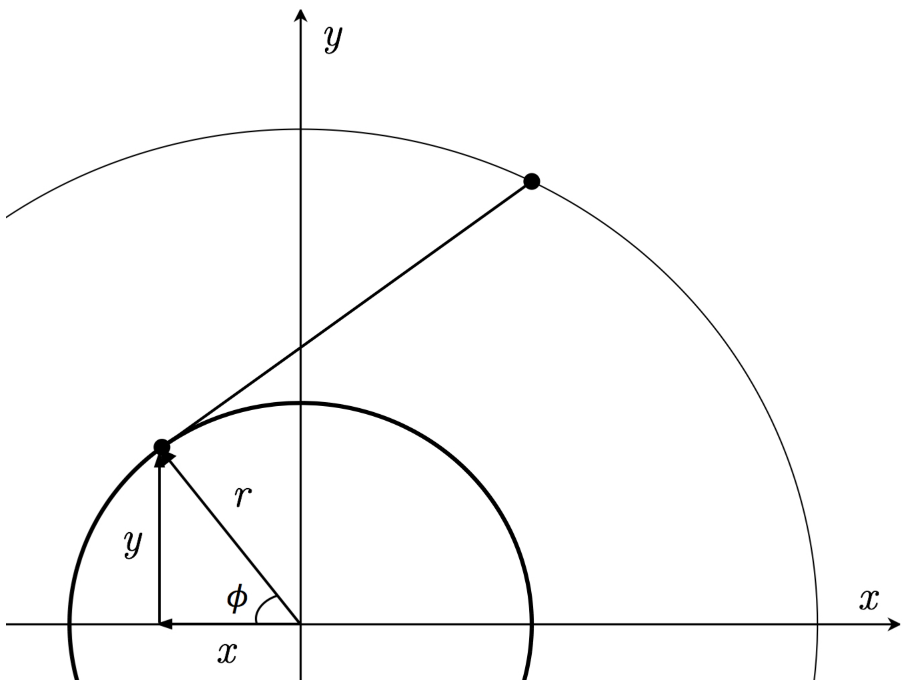
Now, we have everything that we require. However, the objective is to find the **maximum** probability expression. In other words, we have to maximize θ. Again, it is important to reiterate that the ball is travelling clockwise. If we look at the goal post at (x_1, y_1) as the left extreme, the point on the original circle that will make a tangent line with the post will always be in Quadrant II. This is because the question states $y_1 > r$. For the right extreme, point (x_B, y_B), we have a different story.

Although the goal posts are restricted by certain rules, they are not fixed in place. As a result, we can imagine three different possible cases as a result of the position of the posts:

Case 1: $y_B > r$
Case 2: $y_B = r$
Case 3: $y_B < r$

For Case 1, if (x_B, y_B) is situated such that its height is above the circle, the string attached to the ball must snap somewhere in Quadrant II, similar to the left goal post. For Case 2, if (x_B, y_B) is situated such that its height perfectly matches the top apex of the circle, then the string must snap on the y-axis. For Case 3, if (x_B, y_B) is situated such that its height is lower than the radius of the circle, the string must snap somewhere in Quadrant I so that it has a somewhat downward trajectory to reach the point. When we consider all these cases, the only case that gives us the maximum θ, and thus maximum probability, is Case 3 — in this case, the arc on the original circle that permits the trajectory of the ball to score a goal expands from Quadrant II to Quadrant I.

So, what do we have so far? We have (1) described the bound circle and calculated R, (2) derived a formula to calculate the x-coordinates on a circle such that its tangents go through a specific exterior point, and (3) derived a formula to calculate the x-coordinate of the intersection between the goal line and the bound circle. Knowing all these, how do we find angle θ? We can compute this angle by using simple trigonometry. We will need to find out the reference angle of each point on the original circle; let's refer to the diagram below on how to translate an x-coordinate to an angle for some test point.



Note that we only care about the reference angle in this case. Since we already know that the left extreme will be in Quadrant II and the right extreme will be in Quadrant I, we can fix for the actual angle we are interested in (the angle θ). To translate the x-coordinate to an angle, we can use:

$$\cos\phi = \frac{x}{r}$$
$$\phi = \arccos\left(\frac{x}{r}\right)$$

Let's call the angle for the left extreme ϕ_ℓ and the angle for the right extreme ϕ_r,

$$\phi_\ell = \pi - \arccos\left(\frac{x_\ell}{r}\right)$$

$$\phi_r = \arccos\left(\frac{x_r}{r}\right)$$

where x_ℓ corresponds to the point on the circle that its tangent goes through the left extreme, and x_r corresponds to the point on the circle that its tangent goes through the right extreme. Therefore, the probability will be:

$$\begin{aligned} P &= \frac{\phi_\ell - \phi_r}{2\pi} \\ &= \frac{\pi - \arccos\left(\frac{x_\ell}{r}\right) - \arccos\left(\frac{x_r}{r}\right)}{2\pi} \end{aligned}$$

To find x_ℓ and x_r, we have to use our formula that we derived in part (b). Since x_ℓ must be in Quadrant II, we must pick the negative root to get a negative solution. For x_r it will be tricky since both roots give us a positive solution, so which one should we use? Recall that x_r is the point on the circle that its tangent goes through (x_B, y_B). Since (x_B, y_B) is on the arc of the bound circle that is in Quadrant I, we must pick the tangent point that is closer to zero, namely the negative root. This is because the ball is rotating clockwise. If we pick the positive root, we get the tangent point solution that has a higher value for the x-coordinate which actually falls below the original circle. This positive solution would be the one to pick if the ball was rotating counterclockwise. Therefore,

$$x_\ell = \frac{r^2 x_1 - r y_1 \sqrt{x_1^2 + y_1^2 - r^2}}{x_1^2 + y_1^2}$$

$$\begin{aligned} x_r &= \frac{r^2 x_B - r y_B \sqrt{x_B^2 + y_B^2 - r^2}}{x_B^2 + y_B^2} \\ &= \frac{r^2 x_B - r\big(m(x_B - x_1) + y_1\big)\sqrt{R^2 - r^2}}{R^2} \end{aligned}$$

since (x_B, y_B) falls on the bound circle; y_B has been substituted with its equivalent via the linear equation describing the goal line (referenced earlier).

Finally, we have reached an answer. The expression for maximum probability the ball has of scoring a goal, if the string was to snap at a random time, is equivalent to:

$$P = \frac{1}{2\pi}\left[\pi - \arccos\left(\frac{rx_1 - y_1\sqrt{x_1^2 + y_1^2 - r^2}}{x_1^2 + y_1^2}\right) - \arccos\left(\frac{rx_B - \big(m(x_B - x_1) + y_1\big)\sqrt{R^2 - r^2}}{R^2}\right)\right]$$

where

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$R = r\sqrt{\frac{4\pi^4 r^2}{\mu^2 g^2 T^4} + 1}$$

$$x_B = \frac{m(mx_1 - y_1) + \sqrt{R^2(m^2+1) - (mx_1 - y_1)^2}}{m^2 + 1}$$

Of course, there are multiple variations of this answer. For example, you could pick point $P_2(x_2, y_2)$ to work with rather than $P_1(x_1, y_1)$. However, all variations are equivalent.