Solutions to Problem Set No. 4

UBC Metro Vancouver Physics Circle 2018

April 19, 2018

Problem 1 — Balancing with Geometry

Partition the shape into two zones: the rectangle, and the right-angle triangles. The pivot point is the tip of the removed isosceles triangle. The two right-angle triangles form an isosceles triangle equal in area to the removed segment. Due to its symmetry, this problem is equivalent to inverting the triangle regions. Therefore, to balance the torques on both sides, we calculate:

$$\tau_{\text{rectangle}} = \tau_{\text{triangle}}$$

To do so, we must know where the centre-of-mass of the rectangle and the centre-of-mass of the isosceles triangle are located. Since the material is of uniform density, the torques are proportional to the areas of each shape. Let a be the width of the square, h be the height of the triangle, and $a - h$ be the height of the rectangle. The centre-of-mass of an isosceles triangle is located at one-third its height from its base. Relative to the pivot, the centre-of-mass for the isosceles triangle is $\frac{2}{3}h$. It is visually intuitive that the centre-of-mass of the rectangle is located at $\frac{1}{2}(a - h)$.

$$\tau_{\text{rectangle}} = \tau_{\text{triangle}}$$

$$\rho \cdot \left(\frac{1}{2}ah\right)\left(\frac{2}{3}h\right) = \rho \cdot \Big(a(a-h)\Big)\left(\frac{1}{2}(a-h)\right)$$

$$\frac{1}{3}h^2 = \frac{1}{2}(a-h)^2$$

$$0 = 3a^2 - 6ah + 2h^2$$

$$h = a\left(\frac{3 \pm \sqrt{3}}{2}\right)$$

Algebraically, we get two solutions. However, the plus sign solution makes no sense because it would imply that the pivot is located outside the plate. Therefore, the height of the isosceles triangle must be

$$h = a\left(\frac{3 - \sqrt{3}}{2}\right)$$

Problem 2 — Ocean Surface

The fraction of water molecules on the surface f is the ratio of the size of a water molecule l_w to the average depth of the ocean l_d:

$$f = \frac{l_w}{l_d} = \frac{(V_w)^{\frac{1}{3}}}{l_d}$$

where V_w is the volume of a water molecule. To see why this is true, imagine a single column of water molecules extending from the ocean floor to the surface, and the fraction at the surface is simply 1 divided by the number of water molecules in the column.

The volume of a water molecule is simply 1 over the density of water (measured in molecules per cubic metre), so:

$$V_w = \frac{1}{\frac{55 \text{ mol H}_2\text{O}}{\text{L}} \times \frac{6.0 \times 10^{23} \text{ molecules}}{\text{mol}} \times \frac{1000 \text{ L}}{\text{m}^3}} = 3.0 \times 10^{-29} \text{ m}^3$$

The fraction of molecules on the surface is therefore:

$$f = \frac{(3.0 \times 10^{-29} \text{ m}^3)^{\frac{1}{3}}}{3.6 \times 10^3 \text{ m}} = 8.6 \times 10^{-14}$$

Problem 3 — Heating Metal Spheres

When heated, both balls will undergo thermal expansion, which will change the centre of mass of each ball by $\pm\Delta h$ ($+\Delta h$ for ball A, $-\Delta h$ for ball B). By conservation of energy:

$$Q = \Delta E_{\text{thermal}} + \Delta E_{\text{potential}} = mC\Delta T \pm mg\Delta h$$

Solving for the change in temperature gives:

$$\Delta T = \frac{1}{mC}\Big(Q \mp mg\Delta h\Big)$$

Since the second term is negative for ball A but positive for ball B, ball B will have a higher temperature.

Problem 4 — Lagrange Points

(a) Write Newton's 2nd Law for m:

$$\frac{GMm}{R^2} = m \cdot R\omega^2$$

Where $\dfrac{GMm}{R^2}$ is the net force acting on m and $R\omega^2$ is the centripetal acceleration.

$$\omega^2 = \frac{GM}{R^3}$$

$$\therefore \omega = \sqrt{\frac{GM}{R^3}}$$

(b) Write Newton's 2nd Law for μ:

$$-\frac{GM\mu}{r^2} \pm \frac{Gm\mu}{(R \mp r)^2} = -\mu r\omega^2$$

The gravitational force between m and μ is positive when μ is positioned in between M and m, and negative when it's positioned outside of that interval. As for the distance $R \mp r$, it will be negative when μ is positioned between M and m, and positive when it's positioned outside of that interval. In other words, the top sign corresponds to μ being between M and

m, and the bottom sign corresponds to it being outside of that range.

Let us now substitute for ω:

$$-\frac{GM\mu}{r^2} \pm \frac{Gm\mu}{(R \mp r)^2} = -\frac{GM\mu r}{R^3}$$

Divide all terms by $\dfrac{GM\mu r}{R^2}$:

$$-\frac{R^2}{r^3} \pm \frac{mR^2}{Mr(R \mp r)^2} = -\frac{1}{R}$$

Multiply all terms by r, and then factor R out of the $(R \mp r)^2$ term:

$$-\frac{R^2}{r^2} \pm \frac{m}{M\left(1 \mp \dfrac{r}{R}\right)^2} = -\frac{r}{R}$$

Using substitutions $a = \dfrac{m}{M}$ and $x = \dfrac{r}{R}$, we obtain:

$$-\frac{1}{x^2} \pm \frac{a}{(1 \mp x)^2} = -x$$

Finally, multiply by $-x^2$ to get:

$$x^3 = 1 \mp \frac{ax^2}{(1 \mp x)^2}$$

(c) We look at F_{net} exerted on μ from the viewpoint of a rotating frame of reference positioned at M. There is a fictitious centrifugal force equal to $\mu r\omega^2$ in the $+\hat{r}$ direction. Let's label the forces acting on μ:

$$F_M = \frac{GM\mu}{r^2} \qquad F_m = \frac{Gm\mu}{(R \pm r)^2} \qquad F_C = \mu r\omega^2$$

Left side of M:

$$\lim_{r\to\infty} F_M = 0 \qquad \lim_{r\to 0^+} F_M = \infty$$

$$\lim_{r\to\infty} F_m = 0 \qquad \lim_{r\to 0^+} F_m = \frac{Gm\mu}{R^2}$$

$$\lim_{r\to\infty} F_C = \infty \qquad \lim_{r\to 0^+} F_C = 0$$

F_{net} points to the left. $\qquad$ F_{net} points to the right.

Since F_{net} is a continuous function for $r \in (0, \infty)$ and since it has switched direction in this interval, it must have passed zero at some point r_1: $\quad F_{\text{net}}\Big|_{r_1} = 0$

In fact, since the effects of F_m are negligible ($F_m \ll F_M$), it is true that $r_1 \approx R$. Same answer can be obtained from the equation in part (b) $\to$ ($x \approx 1$).

Between m and M:

$$\lim_{r\to 0^+} F_M = 0 \qquad \lim_{r\to R^-} F_M = \frac{Gm\mu}{R^2}$$

$$\lim_{r\to 0^+} F_m = \frac{Gm\mu}{R^2} \qquad \lim_{r\to R^-} F_m = \infty$$

$$\lim_{r\to 0^+} F_C = 0 \qquad \lim_{r\to R^-} F_C = \mu R\omega^2$$

F_{net} points to the left. $\qquad$ F_{net} points to the right.

By similar reasoning as before, there exists an $r_2 \in (0, R)$ such that: $\quad F_{\text{net}}\Big|_{r_2} = 0$

r_2 is also close to R and $r_2 < R$.

Right side of m:

$$\lim_{r\to R^+} F_M = \frac{Gm\mu}{R^2} \qquad \lim_{r\to\infty} F_M = 0$$

$$\lim_{r\to R^+} F_m = \infty \qquad \lim_{r\to\infty} F_m = 0$$

$$\lim_{r\to R^+} F_C = \mu R\omega^2 \qquad \lim_{r\to\infty} F_C = \infty$$

F_{net} points to the left. F_{net} points to the right.

There exists an $r_3 \in (R, \infty)$ such that: $F_{\text{net}}\Big|_{r_3} = 0$
r_3 is also close to R but $r_3 > R$.

Therefore, there are a total of 3 Lagrange points on the line connecting m and M.

(d) From part (b), we have:

$$x_3^3 = 1 + \frac{ax_3^2}{(1-x_3)^2}$$

If $m = 0$, then $a = 0$, and thus $x_3^3 = 1$ solves for $x_3 = 1$. This means that μ would be in the same location as m.

For the case $a \ll 1$:

$$x_3 = 1 + \delta x_3 \qquad \text{(Eqn. 1)}$$

where $\delta x_3 \ll 1$ since x_3 changes by a small amount as well.

$$\therefore (1+\delta x_3)^3 = 1 + \frac{a(1+\delta x_3)^2}{(\delta x_3)^2}$$

Applying the numerical approximation yields:

$$1 + 3\delta x_3 \approx 1 + \frac{a(1+2\delta x_3)}{(\delta x_3)^2}$$

$$(\delta x_3)^2 + 3(\delta x_3)^3 = (\delta x_3)^2 + a + 2a\delta x_3$$

$$3(\delta x_3)^3 = a \qquad \text{(since } a\delta x_3 \ll a\text{)}$$

$$\delta x_3 = \sqrt[3]{\frac{a}{3}}$$

$$\therefore x_3 = 1 + \sqrt[3]{\frac{a}{3}} \qquad \text{(using Eqn. 1)}$$

(e)

$$x_3 = \frac{r_3}{R} = 1 + \sqrt[3]{\frac{a}{3}} \quad \rightarrow \quad r_3 = R + R\sqrt[3]{\frac{a}{3}}$$

$$R\sqrt[3]{\frac{a}{3}} = \Delta r_3 \simeq 1.5 \times 10^9 \text{ m} = 1.5 \text{ million km}$$

$$\frac{\Delta r_3}{r_{\text{moon}}} \simeq 3.9 \text{ times}$$

(f) Let us first write F_{net}:

$$F_{\text{net}} = \mu r \omega^2 - \frac{Gm\mu}{(r-R)^2} - \frac{GM\mu}{r^2}$$

It is important to note that $\omega = \sqrt{\frac{GM}{R^3}}$ no longer applies here. We will instead use conservation of angular momentum to find $\omega(r)$.

$$\mu(r_3)^2\omega_0 = \mu r^2 \cdot \omega(r)$$

$$\therefore \omega(r) = \frac{(r_3)^2\omega_0}{r^2} \quad , \quad \omega_0 = \sqrt{\frac{GM}{R^3}}$$

$$\begin{aligned} F_{\text{net}} &= \mu r \cdot \frac{(r_3)^4(\omega_0)^2}{r^2} - \frac{Gm\mu}{(r-R)^2} - \frac{GM\mu}{r^2} \quad , \quad r_3 = Rx_3 \\ &= \frac{GM\mu}{R^3} \cdot \frac{R^4(x_3)^4}{r^3} - \frac{Gm\mu}{(r-R)^2} - \frac{GM\mu}{r^2} \quad , \quad x = \frac{r}{R} \text{ and } a = \frac{m}{M} \\ &= \frac{GM\mu}{R^2}\left[\frac{(x_3)^4}{x^3} - \frac{a}{(x-1)^2} - \frac{1}{x^2}\right] \end{aligned}$$

Note that plugging $x = x_3$ in the above equation gives $F_{\text{net}} = 0$. However, we will consider a small disturbance to the satellite, hence using $x = x_3 + \delta x$, where $\delta x \ll x_3$:

$$\frac{F_{\text{net}}}{\frac{GM\mu}{R^2}} \equiv \delta f_{\delta x} = \frac{(x_3)^4}{(x_3+\delta x)^3} - \frac{a}{(x_3+\delta x-1)^2} - \frac{1}{(x_3+\delta x)^2}$$

Using $x_3 = 1 + \delta x_3$ from part (d):

$$\delta f_{\delta x} = \frac{x_3}{\left(1 + \frac{\delta x}{x_3}\right)^3} - \frac{a}{(\delta x_3 + \delta x)^2} - \frac{1}{(x_3)^2 \left(1 + \frac{\delta x}{x_3}\right)^2}$$

$$= \frac{x_3}{\left(1 + \frac{\delta x}{x_3}\right)^3} - \frac{a}{(\delta x_3)^2 \left(1 + \frac{\delta x}{\delta x_3}\right)^2} - \frac{1}{(x_3)^2 \left(1 + \frac{\delta x}{x_3}\right)^2}$$

We will now apply numerical approximations. Note that both $\frac{\delta x}{x_3} \ll 1$ and $\frac{\delta x}{\delta x_3} \ll 1$:

$$\delta f_{\delta x} \approx x_3 \left(1 - \frac{3\delta x}{x_3}\right) - \frac{a}{(\delta x_3)^2} \left(1 - \frac{2\delta x}{\delta x_3}\right) - \frac{1}{(x_3)^2} \left(1 - \frac{2\delta x}{x_3}\right)$$

The term with 0^{th} order of δx add up to zero:

$$\delta f_{\delta x} = \delta x \left(-3 + \frac{2a}{(\delta x_3)^3} + \frac{2}{(x_3)^3}\right)$$

We may use $x_3 = 1$ here since δx_3 has negligible effect; hence, $\frac{2}{(x_3)^3} \approx 2$.

Using $\delta x_3 = \sqrt[3]{\frac{a}{3}}$ from part (d):

$$\delta f_{\delta x} = \delta x \left(-3 + 6 + 2\right) = 5\delta x$$

$$\therefore \frac{\delta f}{\delta x} = 5 > 0$$

This means that for $\delta x > 0$: $\delta f > 0$ and hence $F_{\text{net}} > 0$ further increasing x and r. Similarly for $\delta x < 0$: $\delta f < 0$ and $F_{\text{net}} < 0$ further decreasing x and r. Therefore, this orbit is **unstable**.