Solutions to Problem Set No. 5

UBC Metro Vancouver Physics Circle 2018

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Problem 1 — The Equilibrium Enigma

For an object that is fully submerged in solution, suspended by a spring, and thus is balanced, we have:

$$\sum \vec{F} = m \cdot \vec{a} = 0$$

$$F_G - F_B - F_S = 0$$

Where F_G is the gravitational force, F_B is the buoyancy force, and F_S is the spring force.

$$\begin{aligned} mg - \rho_f g V_o - k\Delta x = 0 \\ mg = \rho_f g V_o + k\Delta x \end{aligned}$$

Where ρ_f is the density of the solution and V_o is the volume of the object; we use the entire volume of the object in the equation since it is fully submerged. k is the spring constant, and Δx is the length of the spring that is expanded in order to stabilize the object. m is the mass of the object; we can use its density, ρ_o, to rewrite the equation as:

$$\rho_o g V_o = \rho_f g V_o + k\Delta x$$

Solving for Δx, we see the formation of a linear function $\Delta x(\rho_o)$.

$$\Delta x = \frac{gV_o(\rho_o - \rho_f)}{k}$$

$$\Delta x = \frac{gV_o}{k}\rho_o - \frac{\rho_f g V_o}{k}$$

This function has slope equivalent to $\frac{gV_o}{k}$ and y-intercept as $-\frac{\rho_f gV_o}{k}$. But since gV_o is constant for all cases, we are interested in the following proportions:

$$\text{slope} \propto \frac{1}{k} \qquad\qquad y\text{-intercept} \propto -\frac{\rho_f}{k}$$

Since slope is proportional to only one variable, it is best to start with cases that have the same slope, namely Case 2 and Case 3. Since Case 2 and Case 3 are parallel, they must have identical spring constants. But when we compare their y-intercepts, we see that Case 3's is relatively more negative. This means that the ratio of its solution density to its spring constant must be higher (and thus more negative). Therefore, Case 2 and Case 3 have the same spring strength, but Case 3 has a solution of higher density. This eliminates answers B and D. Now, if we compare Case 1 and Case 2, we observe they have similar y-intercepts but have different slopes. Since Case 1 has a lower slope, it must have a stronger spring (slope is inversely proportional to spring constant). This eliminates answer C. If their y-intercepts are the same, and y-intercept is proportional to the negative ratio of solution density to spring constant, Case 1 must have a higher solution density than Case 2 (the ratios must be the same and Case 1 must have a higher k value than Case 2). This eliminates E. Therefore, the correct answer is choice A. As a side note, Case 1 and Case 3 must have the same solutions (at least solutions with same density) since their x-intercepts are the same. If we want to find the x-intercept, we find that:

$$\Delta x = \frac{gV_o}{k}\rho_o - \frac{\rho_f gV_o}{k}$$

$$0 = \frac{gV_o}{k}\rho_o - \frac{\rho_f gV_o}{k}$$

$$\rho_f = \rho_o$$

Since Case 1 and Case 2 share similar object density, they must also share similar solution density.

Problem 2 — Spherical Charges: Reloaded

For this problem, we will use conservation of energy – it is important to note that in the beginning and final states, we only have potential energy. The initial state involves zero velocity, so all energy stored is potential. The final state involves the charges being closest together, which is exactly when velocity is equal to zero; therefore, all energy stored is potential in the final state as well. In both cases, however, the total potential energy involves gravitational potential energy (Ω) and electric potential energy (Φ). Therefore, if we set the height to zero at the rod, we get

$$\sum E_i = \sum E_f$$

$$\Omega_i + \Phi_i = \Omega_f + \Phi_f$$

$$2mgh + \frac{kq^2}{L} = -2mgy + \frac{kq^2}{\frac{3}{5}L}$$

where $h = \frac{4}{5}L$ and $y =$ the vertical height below the rod to the level of the spheres. In fact, we can solve for y by using the Pythagorean theorem.

$$\left(\frac{3}{10}L\right)^2 + y^2 = \left(\frac{1}{2}L\right)^2$$

$$y = L\sqrt{\frac{1}{4} - \frac{9}{100}}$$

$$y = \frac{2}{5}L$$

Going back to our energy equation:

$$2mgL\left(\frac{4}{5} + \frac{2}{5}\right) = \frac{kq^2}{L}\left(\frac{5}{3} - 1\right)$$

$$\frac{12}{5}mgL = \frac{2}{3} \cdot \frac{kq^2}{L}$$

$$\frac{12}{5} \cdot \frac{3}{2} = \frac{\frac{kq^2}{L^2}}{mg}$$

$$\frac{18}{5} = \frac{F_e}{F_g}$$

$$\therefore \frac{F_e}{F_g} = 3.6$$

Problem 3 — The Balancing Act

Since the system is in equilibrium, we know that the net force on all objects is zero, so we can write:

$$F_{\text{Net},1} = -Az_1z_2(2a)^b - Az_1z_3(a+x)^b = 0$$

$$F_{\text{Net},2} = Az_2z_1(2a)^b + Az_2z_3(a-x)^b = 0$$

$$F_{\text{Net},3} = Az_3z_1(a+x)^b - Az_3z_2(a-x)^b = 0$$

Equation 3 allows us to cancel Az_3 from both sides to write:

$$z_1(a+x)^b = z_2(a-x)^b$$

which allows us to solve for x:

$$\frac{z_1}{z_2} = \left(\frac{a-x}{a+x}\right)^b$$

$$\left(\frac{z_1}{z_2}\right)^{\frac{1}{b}} = \frac{a-x}{a+x}$$

NOTE: In order for us to be able to take $1/b$ root here, $\frac{z_1}{z_2}$ must be a non-negative number, meaning z_1 and z_2 must have the same sign.

$$(a+x)\left(\frac{z_1}{z_2}\right)^{\frac{1}{b}} = a - x$$

$$a\left(\frac{z_1}{z_2}\right)^{\frac{1}{b}} + x\left(\frac{z_1}{z_2}\right)^{\frac{1}{b}} = a - x$$

$$x\left(1 + \left(\frac{z_1}{z_2}\right)^{\frac{1}{b}}\right) = a\left(1 - \left(\frac{z_1}{z_2}\right)^{\frac{1}{b}}\right)$$

$$x = a \cdot \frac{1 - \left(\frac{z_1}{z_2}\right)^{\frac{1}{b}}}{1 + \left(\frac{z_1}{z_2}\right)^{\frac{1}{b}}}$$

$$x = a \cdot \frac{(z_2)^{\frac{1}{b}} - (z_1)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}$$

We can now use either Equation 1 or 2 to solve for z_3. We will use Equation 1 here:

$$-Az_1z_2(2a)^b - Az_1z_3(a+x)^b = 0$$

$$z_3(a+x)^b = -z_2(2a)^b$$

$$z_3 = -z_2\left(\frac{2a}{a+x}\right)^b$$

$$z_3 = -z_2\left(\frac{2}{1 + \frac{(z_2)^{\frac{1}{b}} - (z_1)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}}\right)^b$$

$$z_3 = -z_2\left(\frac{2}{\frac{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}} + \frac{(z_2)^{\frac{1}{b}} - (z_1)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}}\right)^b$$

$$z_3 = -z_2\left(\frac{2}{2 \cdot \frac{(z_2)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}}\right)^b$$

$$z_3 = -z_2\left(\frac{(z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}}{(z_2)^{\frac{1}{b}}}\right)^b$$

$$z_3 = -\left((z_2)^{\frac{1}{b}} + (z_1)^{\frac{1}{b}}\right)^b$$

Problem 4 — The Cannon Conundrum

To start this problem, we need to use one of the kinematics equations to somehow characterize the speed of the cannonballs. After we have characterized it, we need to generalize it for the n^{th} shot. We can begin with the equation:

$$d = v_0 t + \frac{1}{2}at^2$$

This equation is particularly useful to start with because, vertically, the height/displacement of the cannonballs is zero; this is because the platforms are at identical levels. Therefore, we can conclude that the following equation will hold in our vertical dimension:

$$\frac{2v_0}{a} = t$$

If we take into account the first shot, the time (t_1) that corresponds to the time the cannonball spends in the air will be:

$$t_1 = \frac{2v_1 \sin\theta}{g}$$

And if we take into account the second shot, the time (t_2) that corresponds to the time the cannonball spends in the air will be:

$$t_2 = \frac{2v_2 \sin\theta}{g}$$

This is essentially true because the angle is not changing. Therefore, as the second platform approaches, the time the cannonballs spend in the air will become less and less, and, proportionally, the muzzle speed will also decrease. Therefore, we can conclude that the following equation holds for the n^{th} shot:

$$t_n = \frac{2v_n \sin\theta}{g} \qquad \text{(Equation 1)}$$

Now that we have somehow generalized v_n, this is in terms of t_n only, a quantity which is unknown. However, since time is a non-vector variable, it will be true for all dimensions; this makes it a perfect bridge between the vertical and horizontal dimensions. Let's now characterize the muzzle speed in the horizontal direction.

Since there is no acceleration in the horizontal direction, the only kinematics equation we will need to work with is $d = vt$, where d is the horizontal displacement, v is horizontal speed, and t is time. In the first shot, the cannonball will NOT travel a total distance of D, but something less than that. This is true because the second platform is moving; in fact, it is moving with a constant speed. Therefore, the horizontal distance the cannonball actually travels for the first shot will be $D - ut_1$, where t_1 is the time the cannonball from the first shot spends in the air; the term ut_1 corresponds to the distance the second platform has moved in that given time. Therefore, our horizontal equation $d = vt$ for the first shot becomes:

$$D - ut_1 = v_1 \cos(\theta) t_1$$

And similarly, the horizontal equation for the second shot becomes:

$$D - ut_1 - ut_2 = v_2 \cos(\theta) t_2$$

where the term ut_2 corresponds to the horizontal distance the second platform has moved during the second shot. If we generalize this, we get

$$D - ut_1 - ut_2 - ut_3 - ... - ut_{n-1} - ut_n = v_n \cos(\theta) t_n$$

This can be written in summation notation as:

$$D - u \sum_{i=1}^{n} t_i = v_n \cos(\theta) t_n$$

Replacing for t_i and t_n using Equation 1 achieves:

$$D - u \sum_{i=1}^{n} \frac{2v_i \sin\theta}{g} = v_n \cos(\theta) \left(\frac{2v_n \sin\theta}{g} \right)$$

$$D - \frac{2u \sin\theta}{g} \sum_{i=1}^{n} v_i = \frac{v_n^2 \cdot 2 \sin\theta \cos\theta}{g}$$

$$Dg - 2u \sin\theta \sum_{i=1}^{n} v_i = v_n^2 \sin 2\theta \qquad \text{(Equation 2)}$$

This equation will also hold for muzzle speed up to $(n+1)^{\text{th}}$ shot:

$$Dg - 2u\sin\theta \sum_{i=1}^{n+1} v_i = v_{n+1}^2 \sin 2\theta$$

We will pull the $(n+1)^{\text{th}}$ term out of the summation to get:

$$Dg - 2u\sin\theta \sum_{i=1}^{n} v_i - 2uv_{n+1}\sin\theta = v_{n+1}^2 \sin 2\theta$$

Using Equation 2, we get:

$$v_n^2 \sin 2\theta - 2uv_{n+1}\sin\theta = v_{n+1}^2 \sin 2\theta$$

We can now simplify for v_{n+1} to get a quadratic equation, which we can use the quadratic formula to solve for:

$$\begin{aligned}
v_n^2 - uv_{n+1}\sec\theta = v_{n+1}^2 & \\
0 &= v_{n+1}^2 + uv_{n+1}\sec\theta - v_n^2 \\
v_{n+1} &= \frac{-u\sec\theta \pm \sqrt{u^2\sec^2\theta + 4v_n^2}}{2} \\
\therefore v_{n+1} &= \frac{\sqrt{u^2\sec^2\theta + 4v_n^2} - u\sec\theta}{2}
\end{aligned}$$

We must pick the positive solution since that is the only way v_{n+1} can be positive.