Solutions to Problem Set No. 1

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Cannon InfinyX

This question is essentially aimed at finding R, the horizontal displacement of the cannonball from the muzzle of the cannon to the location of landing. In this case, R is simply:

$$R = v\cos(\theta)\,T$$

where T is the time that the cannonball spends in the air from when it is shot out from the muzzle. In order to solve this problem, therefore, we must find v and T since $cos(\theta)$ is already in terms of θ. To solve for the muzzle velocity of the cannonball, we can use one of the primary kinematics equations

$$v = v_0 + at$$

Since $v_0 = 0$ and $a = g$, we get

$$v = g\tau$$

as the ball is accelerated constantly for τ seconds. Next, let us calculate d which is the length of the barrel with respect to τ.

$$\begin{aligned} d &= v_0 t + \frac{1}{2}at^2 \\ &= (0)\tau + \frac{1}{2}g\tau^2 \\ &= \frac{1}{2}g\tau^2 \end{aligned}$$

As a result, the magnitude of the vertical displacement of the cannonball from the muzzle of the cannon to the location of landing is

$$\left|\vec{d_y}\right| = \frac{1}{2}g\tau^2 \sin(\theta)$$

We can now use our other primary kinematics equation to solve for T by picking the downward $\hat{y}$ to be positive:

$$\begin{aligned}
d &= v_0 t + \frac{1}{2}at^2 \\
\frac{1}{2}g\tau^2 \sin(\theta) &= -v \sin(\theta) T + \frac{1}{2}gT^2 \\
\frac{1}{2}g\tau^2 \sin(\theta) &= -g\tau \sin(\theta) T + \frac{1}{2}gT^2 \\
\frac{1}{2}\tau^2 \sin(\theta) &= -\tau \sin(\theta) T + \frac{1}{2}T^2 \\
0 &= \frac{1}{2}T^2 - \tau \sin(\theta) T - \frac{1}{2}\tau^2 \sin(\theta)
\end{aligned}$$

By using the quadratic equation:

$$\begin{aligned}
T &= \frac{\tau \sin(\theta) \pm \sqrt{\tau^2 \sin^2(\theta) + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\tau^2 \sin(\theta)\right)}}{2\left(\frac{1}{2}\right)} \\
&= \tau \sin(\theta) + \sqrt{\tau^2 \sin^2(\theta) + \tau^2 \sin(\theta)} \\
&= \tau\left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right)
\end{aligned}$$

since we must pick the positive solution for T to be positive. We can now solve for R by using our answers for v and T.

$$\begin{aligned}
R &= v \cos(\theta)\, T \\
&= g\tau \cos(\theta) \cdot \tau \left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right) \\
&= g\tau^2 \cos(\theta) \left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right)
\end{aligned}$$

$$= 2d\cos(\theta)\left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right)$$

$$\therefore C(\theta) = 2\cos(\theta)\left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right)$$

We have found our solution for $C(\theta)$, but to make our answer a bit more tidy, we can do the following:

$$\begin{aligned} C(\theta) &= 2\cos(\theta)\left(\sin(\theta) + \sqrt{\sin^2(\theta) + \sin(\theta)}\right) \\ &= 2\cos(\theta)\left(\sin(\theta) + \sqrt{\sin^2(\theta)\left(1 + \frac{1}{\sin(\theta)}\right)}\right) \\ &= 2\cos(\theta)\left(\sin(\theta) + \sin(\theta)\sqrt{1 + \csc(\theta)}\right) \\ &= 2\sin(\theta)\cos(\theta)\left(1 + \sqrt{1 + \csc(\theta)}\right) \end{aligned}$$

$$\therefore C(\theta) = \sin(2\theta)\left(1 + \sqrt{1 + \csc(\theta)}\right)$$

Problem 2 — Colliding Black Holes

a) A black hole, by definition, is a gravitational trap for light. It will therefore involve Newton's constant G, which is related to the strength of gravity, and the speed of light c. The mass of the particle is also relevant, since we expect a heavier particle to correspond to a heavier black hole. We denote the units of a quantity by square brackets, $[\cdot]$. Obviously, $[M] =$ mass and $[c] =$ distance/time. From Newton's law of gravitation,

$$F = \frac{GMm}{r^2} \quad \Longrightarrow \quad [G] = \frac{[F][r]^2}{[M]^2} = \frac{\text{length}^3}{\text{time}^2 \cdot \text{mass}},$$

where we used

$$[F] = [ma] = \text{mass} \cdot \frac{\text{length}}{\text{time}^2}.$$

Area has the units of length^2. We can systematically analyse the units using simultaneous equations, but here is a shortcut: time doesn't appear in the final answer, so we must combine

G and c as G/c^2, which has units

$$[Gc^{-2}] = \frac{\text{length}}{\text{mass}}.$$

To get something with units length^2, we must square this and multiply by M^2. It follows that, up to some dimensionless constant η, the area of the black hole is

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2.$$

b) Consider two black holes of mass M_1, M_2. The initial and final area are

$$A_{\text{init}} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2), \quad A_{\text{final}} = \left(\frac{\eta G^2}{c^4}\right) M_{\text{final}}^2.$$

If $A_{\text{init}} = A_{\text{final}}$, we have maximal loss of mass; if $M_{\text{final}} = M_1 + M_2$, we minimise the mass loss. The percentage of mass lost will depend on the mass of the black holes, but to place an upper bound, we want to choose the masses to maximise the fraction of mass lost. The simplest way to proceed is to instead look at the difference of *squared* masses,

$$\Delta M^2 = M_{\text{final}}^2 - M_1^2 - M_2^2 = (M_1 + M_2^2)^2 - M_1^2 - M_2^2 = 2M_1M_2.$$

Since we only care about the fraction lost, we can require a total initial mass $M = M_1 + M_2$ for fixed M, and now try to choose M_1, M_2 to maximise the square of mass lost:

$$\Delta M^2 = 2M_1M_2 = 2M_1(M - M_1).$$

This is just a quadratic in M_1, with roots at $M_1 = 0$ and $M_1 = M$. The *maximum* will be precisely in between, at $M_1 = M/2$. Of course, maximising the square of lost mass should be the same as maximising the lost mass itself, so we obtain an upper bound on mass loss in any black hole collision by setting $M_1 = M_2$, with a fractional loss

$$1 - \frac{M_{\text{final}}}{M_1 + M_2} = 1 - \frac{\sqrt{M_1^2 + M_1^2}}{M_1 + M_1} = 1 - \frac{\sqrt{2}}{2} \approx 0.29.$$

Since the mass can be converted into gravitational waves, we have the 29% bound we were looking for!

c) From the last question, we know that we maximise the energy converted into gravitational waves when the total area doesn't change,

$$A_{\text{final}} = A_1 + A_2 = \frac{\eta G^2}{c^4}(M_1^2 + M_2^2) = \left(\frac{\eta G^2}{c^4}\right) M_{\text{final}}^2.$$

This corresponds to a loss of mass

$$\Delta M = M_1 + M_2 - M_{\text{final}} = M_1 + M_2 - \sqrt{M_1^2 + M_2^2} \approx 18.9\, M_\odot.$$

We can convert this to energy using the most famous formula in physics, $E = mc^2$. To find the average power P, we divide by the duration of the signal $t = 0.2$ s. We find

$$P_{\text{BH}} = \frac{E}{t} = \frac{\Delta M c^2}{t} = \frac{18.9 \cdot 2 \cdot 10^{30} (3 \times 10^8)^2}{0.2}\ \text{W} \approx 1.7 \times 10^{49}\ \text{W}.$$

Since $P_{\text{BH}} > P_{\text{stars}}$, we see that for a brief moment, colliding black holes can outshine all the stars in the universe.

Problem 3 — Charges on a Rail

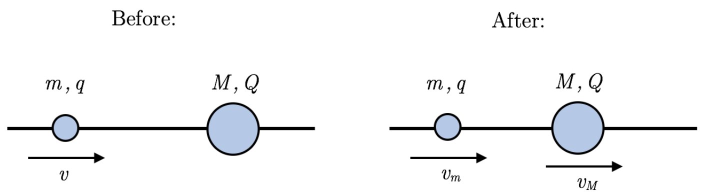
a) Write energy conservation for the system:

$$\frac{mv^2}{2} = \frac{kqQ}{d_f} \quad \longrightarrow \quad d_f = \frac{2kqQ}{mv^2}$$

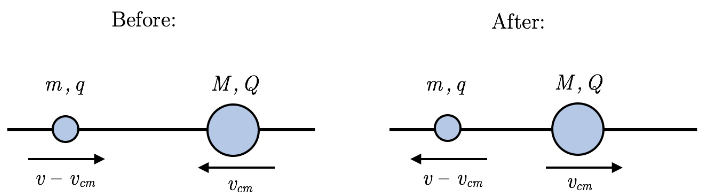
b) One way of solving would be to write energy and momentum conversation for the system and solve equations to find v_m and v_M:

$$\begin{aligned} E: &\qquad \frac{mv^2}{2} = \frac{mv_m^2}{2} + \frac{Mv_M^2}{2} \\ P: &\qquad mv = mv_m + Mv_M \end{aligned}$$

An easier and more elegant method is to view the system from the perspective of centre of momentum. From the frame of reference of the stationary observer:



From the frame of reference of centre of momentum moving with constant velocity $v_{cm} = \dfrac{mv}{m+M}$ to the right:



Since momentum of the system is zero in this frame of reference, the collision simply causes a change in direction of velocities as shown above. Therefore, transforming velocities to stationary frame:

$$v_m = 2v_{cm} - v = \frac{(m-M)v}{m+M}$$

$$v_M = 2v_{cm} = \frac{2mv}{m+M}$$

c) Minimum distance d_r is achieved when m and M have equal velocities. Let's see what's the reason why this is the case. We write momentum and energy conservation taking $v_m = v_M = v'$:

$$E: \qquad \frac{mv^2}{2} = \frac{mv'^2}{2} + \frac{Mv'^2}{2} + \frac{kqQ}{d_r}$$

$$P: \qquad mv = mv' + Mv'$$

From momentum, solving for v', we have $v' = \dfrac{mv}{m+M}$ and plugging into energy:

$$\frac{mv^2}{2} = \frac{(m+M)}{2} \cdot \frac{m^2v^2}{(m+M)^2} + \frac{kqQ}{d_r}$$

$$\therefore \quad d_r = \frac{2kqQ(m+M)}{mMv^2}$$

d)

$$\frac{d_r}{d_f} = 1 + \frac{m}{M}$$

In limiting case $\frac{m}{M} \ll 1$, we have $d_r \approx d_f$. This is intuitive since the much larger mass of M means fixing it does not have much effect on the system. In limiting case $\frac{m}{M} \gg 1$, we get $\frac{d_r}{d_f} \gg 1$. This outcome also makes sense since if M is not fixed, it can easily gain momentum and escape due to its low mass.

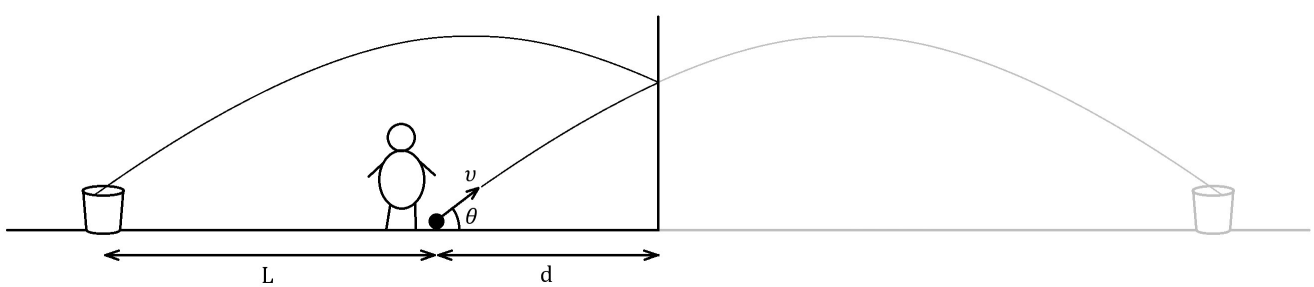
Problem 4 — "Trick-Shot" Tyler

a) Since the ball has a perfectly elastic collision with the wall, the y component of its velocity remains unchanged while the x component of its velocity is reflected.

$$v'_x = -v_x$$

$$v'_y = v_y$$

This means that if we look at the reflection of the path of the ball in the wall (as if the wall was a mirror), we can see an undisturbed projectile motion with range $R = \frac{v^2 \sin 2\theta}{g}$ (try to derive this if it is unfamiliar):



Therefore, we have $R = 2d + L$ and thus:

$$L = \frac{v^2 \sin 2\theta}{g} - 2d$$

b)

$$v_x = v \cos\theta \qquad v_y = v \sin\theta - g t_{\text{collision}}$$

We can find $t_{\text{collision}}$ from motion in x:

$$v_x t_{\text{collision}} = d \quad \longrightarrow \quad t_{\text{collision}} = \frac{d}{v_x}$$

Therefore,

$$v_y = v\sin\theta - \frac{gd}{v_x} = v\left(\sin\theta - \frac{gd}{v^2\cos\theta}\right)$$

c) Since no slipping occurs, we have $v'_y = R\omega'$. And by momentum conservation in x, $v'_x = v_x$. We can write energy conservation:

$$\frac{mv^2}{2} = \frac{m(v')^2}{2} + \frac{I(\omega')^2}{2}$$

$$v^2 = v_x^2 + v_y^2$$

$$(v')^2 = (v'_x)^2 + (v'_y)^2$$

$$= v_x^2 + (v'_y)^2$$

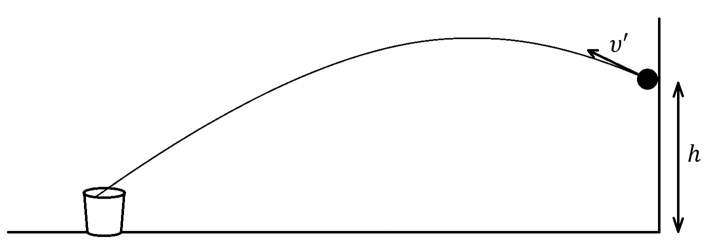
Hence, the energy conservation simplifies to:

$$mv_y^2 = m(v'_y)^2 + I\left(\frac{v'_y}{R}\right)^2$$

$$v_y^2 = (v'_y)^2 + \frac{2}{5}(v'_y)^2$$

$$v'_y = \sqrt{\frac{5}{7}}\, v_y$$

d) Using our answers to parts (b) and (c) we are solving for the range of projectile in the following setup:



Where

$$h = v_y t_{\text{collision}} - \frac{g t_{\text{collision}}^2}{2} = d \tan\theta - \frac{g d^2}{2 v^2 \cos^2(\theta)}$$

We can write equations for x and y:

$$y_T = 0 = h + v_y' T - \frac{g T^2}{2}$$

$$x_T = v_x' T$$

From first equation, we can solve for T:

$$T = \frac{v_y'}{g} + \sqrt{\frac{(v_y')^2}{g^2} + \frac{2h}{g}}$$

And since $L' = x_T - d$, we have:

$$L' = \frac{v_x' v_y'}{g} + \frac{v_x'}{g} \sqrt{(v_y')^2 + 2gh} - d$$

Using the obtained values for v_x' and v_y' we know:

$$\frac{v_x' v_y'}{g} = \sqrt{\frac{5}{7}} \left(\frac{v^2 \sin 2\theta}{2g} - d \right)$$

and defining $C \equiv \dfrac{v^2 \sin 2\theta}{2gd}$, we can simplify:

$$\frac{v_x' v_y'}{g} = \sqrt{\frac{5}{7}}\, d\, (C - 1)$$

Furthermore, we know:

$$\frac{(v_x')^2 h}{g} = \frac{v^2 \cos^2(\theta)}{g} \left(d \tan\theta - \frac{g d^2}{2 v^2 \cos^2(\theta)} \right)$$

$$= \frac{v^2 \sin(2\theta) d}{2g} - \frac{d^2}{2} = d^2 \left(C - \frac{1}{2} \right)$$

Using what we obtained for $\dfrac{v_x' v_y'}{g}$ and $\dfrac{(v_x')^2 h}{g}$, we have:

$$
\begin{aligned}
L' &= \frac{v_x' v_y'}{g} + \sqrt{\left(\frac{v_x' v_y'}{g}\right)^2 + \frac{2(v_x')^2 h}{g}} - d \\
&= \sqrt{\frac{5}{7}}\, d\,(C-1) + \sqrt{\frac{5}{7} d^2 (C-1)^2 + 2d^2 \left(C - \frac{1}{2}\right)^2} - d \\
&= d\left[\sqrt{\frac{5}{7}}(C-1) + \sqrt{\frac{5}{7}(C-1)^2 + 2C - 1} - 1\right]
\end{aligned}
$$

e) Using values provided, we have $C = 2.5$ and therefore:

$$L = 2d(C-1) = 6 \text{ m}$$

$$L' = 5.27 \text{ m}$$

$$\therefore \Delta L = L - L' = 0.73 \text{ m}$$

This is quite a significant difference.