Problem Set No. 1

UBC Metro Vancouver Physics Circle 2018-2019

November 22, 2018

Problem 1 — Cannon InfinyX

The barrel of a cannon is the long chamber in which a cannonball gains speed before being shot out of the muzzle. Let's assume there's a special type of a cannon, depicted in Figure 1, which is known as *Cannon InfinyX*. This particular cannon is able to accelerate a cannonball at a constant value of g m/s^2 from rest, where g is the gravitational acceleration on Earth at sea level, all the way through the muzzle. For the cannon in Figure 1, a cannonball takes τ seconds to exit the muzzle.

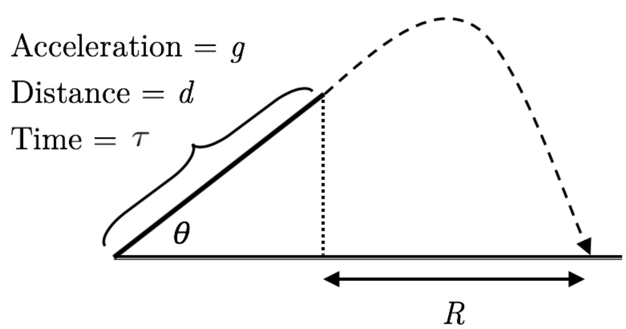


Figure 1: Cannon InfinyX situated at an angle θ from the horizontal. The cannon has barrel length of d metres; the cannonball is shot from rest, accelerated at a constant g m/s^2 for τ seconds.

Let the range, which is the horizontal displacement of the cannonball after it exits the muzzle, be R. If it's true that $R = C \cdot d$, where C is a function of θ, find $C(\theta)$ for $0 < \theta < \pi/2$.

Problem 2 — Colliding Black Holes

When a star runs out of nuclear fuel, it can collapse under its own weight to form a black hole: a region where gravity is so strong that even light is trapped. Black holes were predicted in 1915, but it took until 2015, 100 years later, for the Laser Interferometer Gravitational-wave Observatory (LIGO) to observe them directly. When two black holes collide, they emit a characteristic "chirp" of *gravitational waves* (loosely speaking, ripples in spacetime), and through an extraordinary combination of precision physics and engineering, LIGO was able to hear this chirp billions of light years away.

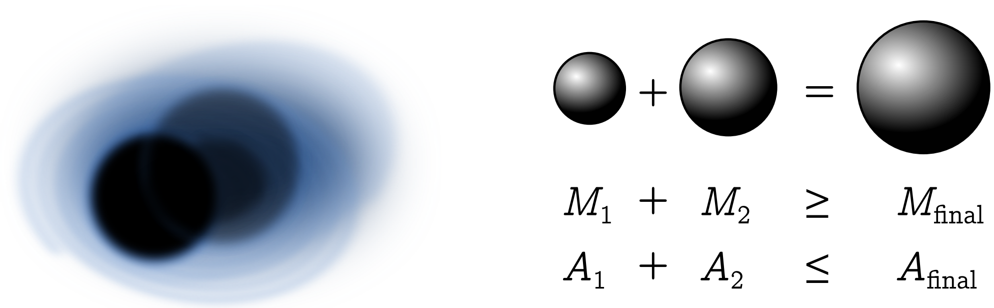


Figure 2: On the left, a cartoon of a black hole merger. On the right, inequalities obeyed by mergers: the mass of the final black hole can decrease when energy is lost (e.g. to gravitational waves), but the area always increases.

a) An infinitely dense point particle of mass M will be shrouded by a black hole. Using dimensional analysis, argue that this black hole has surface area

$$A = \left(\frac{\eta G^2}{c^4}\right) M^2$$

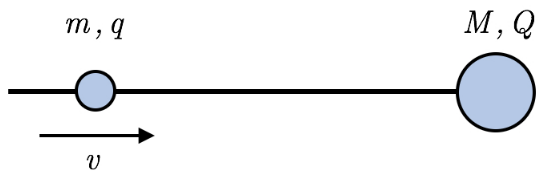
for some constant η.

b) One of Stephen Hawking's famous discoveries is the *area theorem*: the total surface area of any system of black holes increases with time. Using the area theorem, and the result of part (1), show that two colliding black holes can lose at most 29% of their energy to gravitational waves. (Note that to find this upper bound, you need to consider varying the mass of the colliding black holes, and to assume that any lost mass is converted into gravitational waves.)

c) LIGO detected a signal from two black holes smashing into each other 1.5 billion light years away. Their masses were $M_1 = 30M_\odot$ and $M_2 = 35M_\odot$, where $M_\odot \approx 2 \times 10^{30}$ kg is the mass of the sun, and the signal lasted for 0.2 seconds. Assuming the maximum amount of energy is converted into gravitational waves, calculate the average power $P_{\rm BH}$ emitted during the collision. Compare this to the power output of all the stars in the universe, $P_{\rm stars} \sim 10^{49}$ W.

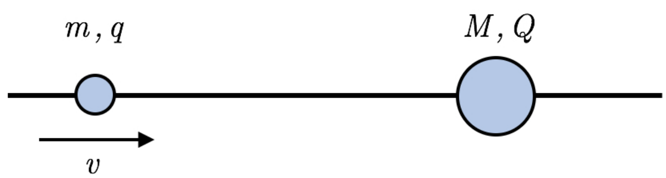
Problem 3 — Charges on a Rail

A particle of mass M and charge Q is fixed in place on a rail. Another particle of mass m and charge q (restricted to move along the rail) is shot towards the first particle with initial velocity v from infinity.



a) Find the minimum distance d_f achieved between the two particles.

Now, we release mass M to move freely as well and repeat the experiment.



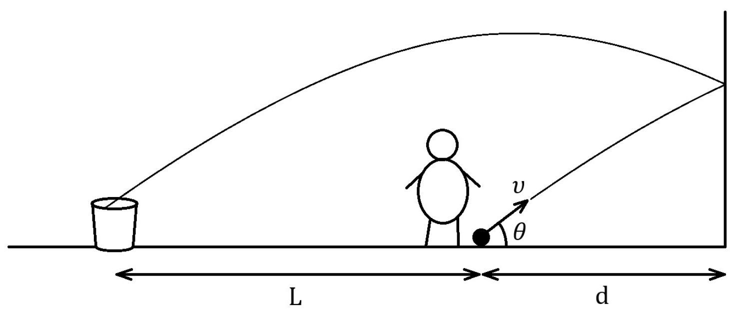
b) Find the velocities of the two particles v_m and v_M.

c) Find the minimum distance d_r achieved between the two particles.

d) Find the ratio d_r/d_f and see what happens in limiting cases $\frac{m}{M} \ll 1$ and $\frac{m}{M} \gg 1$. Explain why the result makes sense.

Problem 4 — "Trick-Shot" Tyler

As an addition to his trick shot compilation, Tyler decides to shoot the ball, bounce it off the wall d meters in front of him and land it in a basket.



a) Provided he shoots the ball with initial velocity v and angle θ from the horizontal, at what L should he place the basket to score? Assume the ball is a point particle.

To his dismay, the ball comes just short of the basket; so, he decides to do more accurate physics calculations.

b) Find the horizontal and vertical components of velocity v_x and v_y with which the ball hits the wall.

Now, assume the ball has radius R and mass m uniformly distributed throughout, and that no slipping occurs when the ball comes in contact with the wall.

c) Find velocity components v'_x and v'_y and angular velocity ω' with which the ball leaves the wall assuming a perfect elastic collision and zero initial angular velocity. (Hint: The moment of inertia of a uniform ball of mass m and radius R is $I = \frac{2}{5}mR^2$.)

d) Find the new position L' where the basket should be placed. (Hint: You might find it helpful to define $C \equiv \dfrac{v^2 \sin(2\theta)}{2gd}$ to simplify calculations.)

e) Find the change in position of basket $\Delta L = L - L'$ for $v = 10$ m/s, $\theta = \pi/4$, $d = 2$ m and $g = 10$ m/s^2.