Solutions to Problem Set No. 3

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Quantum Strings and Vacuums

1. From the picture, we see that λ is an allowed wavelength if L is a multiple of $\lambda/2$. More precisely,

$$L = \frac{n\lambda}{2} \quad \implies \quad \lambda_n = \frac{2L}{n}.$$

2. The total rest energy of the quantum string is

$$E^0 = \frac{\alpha}{\lambda_1} + \frac{\alpha}{\lambda_2} + \frac{\alpha}{\lambda_3} + \cdots = \frac{\alpha}{2L}(1 + 2 + 3 + \cdots) = -\frac{\alpha}{2L} \cdot \frac{1}{12} = -\frac{\alpha}{24L}.$$

3. A jump in energy ΔE in energy over a distance Δx leads to an average force

$$F_{\text{avg}} = -\frac{\Delta E}{\Delta x}.$$

In this case, the distance over which the energy drops is the thickness of the plates, $\Delta x = \ell$, while the change in energy (as we move into the area between plates) is

$$\Delta E = E_{\text{plates}} - E_{\text{vacuum}} = E_{\text{plates}} = \frac{\alpha}{24L},$$

since the energy for the electromagnetic waves between plates takes the same form as harmonics in the stretched string. Thus, the average force on each plate is

$$F_{\text{avg}} = -\frac{E^0}{\ell} = \frac{\alpha}{24\ell L}.$$

This is positive, hence directed *towards* the region between plates. This means the plates are squeezed together!

4. If there are D directions, then one direction is parallel to the string, and the remaining $D-1$ directions are perpendicular to it. Thus, there are $D-1$ independent directions the string can wobble in.

5. There are $D-2$ directions with all harmonics at rest, and one direction with its first harmonic (the red vibration in the picture above) in its first energy level. From question 2, the unexcited directions have total rest energy

$$E_0 = -\frac{\alpha}{24L}.$$

From the expression for E^i_{mn}, we see that by setting $m = n = 1$, we add an energy

$$\frac{2\alpha m}{\lambda_n} = \frac{2\alpha}{\lambda_1} = \frac{\alpha}{L}$$

to the unexcited energy of the harmonic. Thus, the total energy of the string is

$$E = (D-2)E_0 + \left(E_0 + \frac{\alpha}{L}\right) = \frac{\alpha}{L}\left(-\frac{D-1}{24} + 1\right).$$

If the photon is massless, then $m = 0$, and by the most famous formula in physics,

$$E = mc^2 = 0.$$

This implies that

$$-\frac{D-1}{24} + 1 = 0 \quad \Longrightarrow \quad D = 25.$$

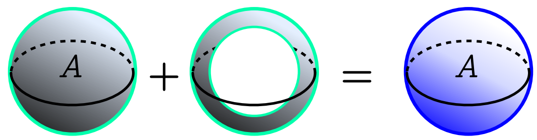
Problem 2 — Black Hole Hard Drives

1. We calculate the entropy from the area law, and convert the answer from bits to GB:

$$\begin{aligned} S &= \frac{A}{A_0}\,\text{bits} \\ &\approx \frac{4\pi(10^{-15})^2\,\text{m}^2}{10^{-69}\,\text{m}^2}\,\text{bits} \\ &\approx 1.25 \times 10^{40}\,\text{bits} \\ &\approx \frac{1.25 \times 10^{40}}{8 \times 10^9}\,\text{GB} \\ &\approx 1.6 \times 10^{30}\,\text{GB} \approx 5 \times 10^{20}\,\text{global computer storage.} \end{aligned}$$

A tiny black hole contains more information than all the world's computers, by an unimaginably large factor $\sim 10^{20}$. That's roughly the number of grains of sand in the world! Perhaps GoogleX is working on black hole hard drives as we speak.

2. First, note that the mass of the sphere M must be smaller than the mass of the corresponding black hole M_A, otherwise it would have already collapsed! We can therefore add a spherical shell of matter, mass $M_A - M$, and compress it to ensure the surface area is A. By assumption, this spherical object will immediately collapse to form a new black hole. Schematically, we are performing the following "sum":



The shell of matter has its own entropy S'', so the total entropy of system before collapse is larger than the black hole entropy:

$$S' + S'' > S' > S_{\text{BH}}.$$

However, after the collapse, the entropy is just the black hole entropy S_{BH}. So we seem to have reduced the total entropy! This violates the Second Law of Thermodynamics. Our assumption, that $S' > S_{\text{BH}}$, must have been incorrect. We learn that black holes are the best spherical hard drives in existence!

3. Black holes have maximum entropy density. Using the area law, the entropy density of a black hole of radius r is

$$\frac{S}{V} = \frac{4\pi r^2}{A_0 4\pi r^3/3} = \frac{3}{A_0 r}.$$

4. The previous result shows that, as a spherical hard drive gets large, the *maximum* information density gets very low. Since this is a maximum, density and hence processing speed is low in *any* large hard drive.

Problem 3 — Swinging on a Swing

3.1 Pumping from the Sitting Position

1. Taking each mass m to represent the center of mass of the upper and lower halves of the body, we have:
$$m = \frac{M}{2}\ ,\ a = \frac{h}{4}$$
And a reasonable angle of rotation for the body would be:
$$\Delta\phi = \frac{\pi}{2}$$

2. At the left-most extreme the barbell should rotate CCW (rider switching from the sat to the leaned back position) to increase the swing amplitude by conservation of angular momentum. In contrast, at the right-most extreme the barbell should rotate CW.

3. Writing the angular momentum conservation about the point of attachment of swing to the ceiling, we have:
$$\Delta L = (Ml^2 + 2ma^2)\,\Delta\theta - 2ma^2\,\Delta\phi$$
whereby $I_o = Ml^2 + 2ma^2$ is the moment of inertia of the swing by the parallel axis theorem. Substituting our approximations from part (1), we get:
$$\Delta\theta = \frac{\Delta\phi \cdot 2\left(\frac{M}{2}\right)a^2}{Ml^2 + 2\left(\frac{M}{2}\right)a^2} = \frac{\Delta\phi}{1 + \frac{l^2}{a^2}} = \frac{\pi/2}{1 + \frac{16l^2}{h^2}}$$

4. Since every half-oscillation of the swing results in an increase in amplitude of $\Delta\theta$, after n half-oscillation we have:
$$\theta_n = \theta_0 + \frac{n\pi/2}{1 + \frac{16l^2}{h^2}}$$
An interesting observation is that this amplitude is independent of the rider?s mass. So your mass does not affect how fast you can pump a swing when seated; more height is however an advantage.

3.2 Pumping from the Standing Position

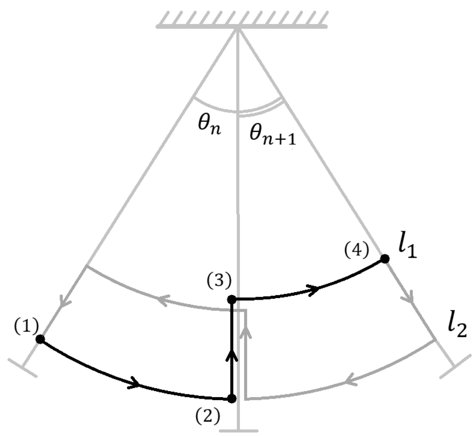
1. While stood, assuming the body?s center of mass is at the center, we have:

$$l_1 = l - \frac{h}{2}$$

And assuming the rider squats to half their height, we have:

$$l_2 = l - \frac{h}{4}$$

2. When moving the fastest, at $\theta = 0$, the rider does work by moving up against the centrifugal force ($2 \to 3$ in the figure below), as perceived in the rotating frame of reference. This work results in an increase in the amplitude.



3. Let us first write the work done by the rider, assuming that the centrifugal force is constant (since $h \ll l$):

$$W = Fd \approx \frac{Mv^2}{l} \cdot \frac{h}{4} = \frac{Mv^2}{2} \cdot \frac{h}{2l}$$

And applying energy conservation between (1) and (2):

$$W = Mgl\left(1 - \cos\left(\theta_n\right)\right) \cdot \frac{h}{2l} = \frac{Mgh\left(1 - \cos\left(\theta_n\right)\right)}{2}$$

Now writing energy conservation between (1) and (4) and noting that the work done has resulted in an overall increase in rider's potential energy:

$$-Mgl_1 \cos(\theta_{n+1}) = -Mgl_2 \cos(\theta_{n+1}) + \frac{Mgh\,(1-\cos(\theta_n))}{2}$$

Let us define $\epsilon \equiv \frac{h}{l} \ll 1$ for convenience. Simplifying the above equation, we get:

$$\cos(\theta_{n+1}) = \left(\frac{1-\epsilon/4}{1-\epsilon/2}\right)\cos(\theta_n) - \frac{\epsilon}{2}(1-\cos(\theta_n))$$

We can make the approximation $\frac{1-\epsilon/4}{1-\epsilon/2} \approx 1+\frac{\epsilon}{4}$ and solve for $\cos(\theta_{n+1})$:

$$\cos(\theta_{n+1}) = \left(1+\frac{3\epsilon}{4}\right)\cos(\theta_n) - \frac{\epsilon}{2}$$

4. Starting with $\cos(\theta_0)$ and applying the relation found in part (3) recursively:

$$\cos(\theta_1) = \left(1+\frac{3\epsilon}{4}\right)\cos(\theta_0) - \frac{\epsilon}{2}$$

$$\cos(\theta_2) = \left(1+\frac{3\epsilon}{4}\right)\left(\left(1+\frac{3\epsilon}{4}\right)\cos(\theta_0) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2}$$

And now applying first-order approximations, we have:

$$\cos(\theta_2) = \left(1+\frac{2\cdot 3\epsilon}{4}\right)\cos(\theta_0) - \frac{2\epsilon}{2}$$

Applying this relation a few more times, we start to see a pattern describing the general solution after n half-oscillations (visit the "Recurrence relation" page on Wikipedia to see more formal and rigorous methods for solving recurrence relations):

$$\cos(\theta_n) = \left(1+\frac{3n\epsilon}{4}\right)\cos(\theta_0) - \frac{n\epsilon}{2} = \cos(\theta_0) + \frac{n\epsilon}{2}\left(\frac{3}{2}\cos(\theta_0) - 1\right)$$

5. We can see that to our approximations, the growth rate for amplitude is linear for (3.1). This is also the case for (3.2) to a reasonable approximation. Both growth rates are independent of rider's mass and solely depend on the ratio $\frac{h}{l}$.
Another observation is that for pumping when seated (3.1), the incremental increases in amplitude are independent of the initial amplitude, while for pumping when standing

(4.1), an initial amplitude of $\theta_0 > \cos^{-1}(2/3) \approx 48^\circ$ is required to pump the swing at all! This is a very unexpected result; it indicates the weakness of our second model. One cause is the assumption $h \ll l$ we made to simplify the problem, which does not hold since with reasonable estimates we have $\frac{h}{l} \approx 0.5$.