Problem Set No. 3

UBC Metro Vancouver Physics Circle 2018-2019

January 24, 2019

Problem 1 — Quantum Strings and Vacuums

Suppose we stretch a string of length L between two fixed points. The string can oscillate sinusoidally in *harmonics*, the first few of which are sketched on the left below. Remarkably, by considering that harmonics of *space itself*, we can show that empty vacuum likes to push metal plates together!

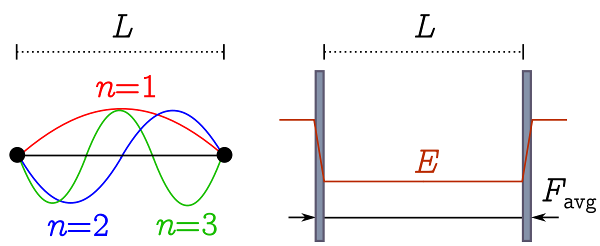


Figure 1: *Left.* Harmonics of a classical string. *Right.* Casimir effect on plates in a vacuum.

1. Show that harmonics on the string have wavelength

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots.$$

2. A classical string can vibrate with some combination of harmonics, including *no harmonics* when the string is at rest. In this case, the string has no energy. A *quantum* string is a little different: even if a harmonic is not active, there is an associated *zero-point energy*:

$$E_{0n} = \frac{\alpha}{\lambda_n},$$

where α is a constant of proportionality. This is related to *Heisenberg's uncertainty principle*, which states that we cannot know both the position and momentum of the string with absolute certainty. Let's calculate the zero-point energy of a quantum string.

Sum up the zero-point energies for each harmonic to find the energy of an unexcited quantum string. Use the infamous result[1] that

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

3. Classical strings can be found everywhere, but where do we find quantum strings? One answer is *space itself*. Instead of stretching a string between anchors, set two lead plates a distance L apart. (Pretend that it vibrates in a plane, as in the picture above.) The harmonics are no longer wobbling modes of the string, but *electromagnetic waves*. Outside the plates is empty space, stretching away infinitely; it has zero energy.[2]

   Suppose that the lead plates have thickness ℓ. Show that the plates are pushed together, with each subject to an average force

$$F_{\text{avg}} = \frac{\alpha}{24\ell L}.$$

   The remarkable fact that the vacuum can exert pressure on parallel metal plates is called the *Casimir effect*. Although weak, it can be experimentally detected!

Bonus. These methods can also be applied to *string theory*. String theory posits that everything in the universe is made out of tiny vibrating strings. Different subatomic particles, like electrons and photons, correspond the different ways that the string can vibrate. We will learn that string theory requires 25 spatial dimensions![3]

When we treat the string as a quantum object, each independent direction gets independent harmonics. Put a different way, we can split the string into $D - 1$ independent strings

[1]There are various ways of proving this, but the basic idea is that very large numbers in this sum correspond to high frequencies which would break the string if we tried to excite them. So we have to throw most of these large numbers away, i.e. subtract them from our running tally. In the process, we overcorrect and get a slightly negative result!

[2]We can model the edge of space with lead plates infinitely far away. Since $L \to \infty$, $E_n^0 \to 0$ and the energy does indeed disappear.

[3]Since we only see three dimensions, the remaining 22 must somehow be "curled up" and hidden from view.

wobbling in two dimensions, labelled by $i = 1, 2, \ldots, D-1$. To get different fundamental particles, we need to be able to *excite* harmonics. It turns out that, according to quantum theory, they have discrete *energy levels*, separated by "quantum leaps" in energy:

$$E^i_{mn} = \frac{\alpha}{\lambda_n}(1+2m), \quad m = 0, 1, 2, \ldots.$$

The superscript i denotes the direction the harmonic wobbles; the subscript n refers to the harmonic, while m refers to how excited that harmonic is. To find the total energy of the string, we just add up the energy of each harmonic.

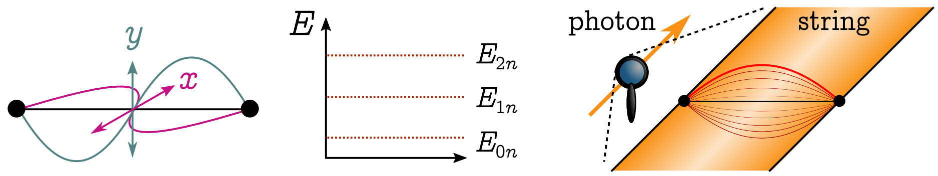


Figure 2: *Left.* Strings vibrating in different independent directions. *Middle.* Discrete quantum energy levels for a particular harmonic. *Right.* If we zoom in on a photon, we get a string with a single excited harmonic.

4. The string can vibrate in any direction perpendicular to the string. In three spatial dimensions, there are two perpendicular directions for the string to vibrate (labelled by x and y above). Explain why, for D spatial dimensions, the string can vibrate in $D-1$ independent directions.

5. Suppose that we excite a first harmonic ($n = 1$) in some direction to its lowest excited state ($m = 1$). A string vibrating this way looks like a *photon* from far away, i.e. a particle of light. Use the fact that the photon *has zero mass* to deduce that $D = 25$. In other words, if string theory is correct, and photons have no mass, then the universe has 25 dimensions!

Problem 2 — Black Hole Hard Drives

Black holes are perhaps the most mysterious objects in the universe. For one, things fall in and never come out again. An apparently featureless black hole could conceal an elephant, the works of Shakespeare, or even another universe! Suppose we wanted to describe all the possible objects that could have fallen into the black hole, but using *binary digits* (bits) 0 and 1, the language of computers. With one bit, we can describe two things, corresponding to 0 and 1; with two bits, we can describe *four* things, corresponding to $00, 01, 10, 11$. Continuing this pattern, with n bits we can describe 2^n things, corresponding to the 2^n sequences of n binary digits. The total number of bits needed to describe all the possibilities, for a given black hole, is called the *entropy* S. Since information is also stored in bits, we can (loosely) equate entropy and information!

We would expect that a large black hole can conceal more than a small black hole, and will therefore have a larger entropy. The *area law*, discovered by Stephen Hawking and Jacob Bekenstein, shows that this is true, with the entropy of the black hole proportional to its *surface area* A:

$$S = \frac{A}{A_0},$$

where $A_0 \approx 10^{-69}\,\mathrm{m}^2$ is a basic unit of area. We can view the black hole surface as a sort of screen, made up of binary pixels of area A_0.

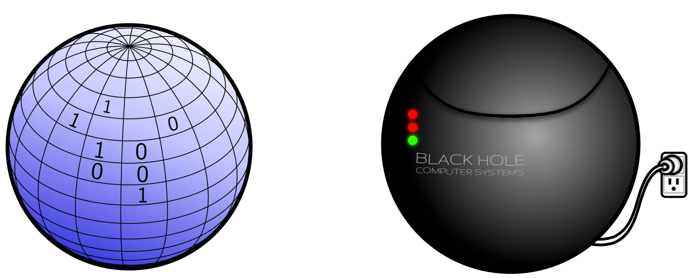


Figure 3: *Left.* The area law, viewed as pixels on the black hole surface. *Right.* A spherical hard drive.

The *Second Law of Thermodynamics* states that the total entropy of a closed system always increases.[4] Combining the area law and the Second Law leads to a surprising conclu-

[4] The entropy of a black hole is the number of bits needed to describe all the things that could have fallen in. The entropy of an ordinary object, like a box of gas, is the number of bits needed to describe all the different *microscopic* configurations which are indistinguishable to a macroscopic experimentalist, i.e. which look like the same box of gas. The function of entropy, in both cases, is to count the number of configurations which look the same!

sion: black holes have the *highest* entropy density of any object in the universe. They are the best hard drives around![5]

1. To get a sense of scale, calculate how many gigabytes of entropy can be stored in a black hole the size of a proton, of radius $\sim 10^{-15}$ m. Note that

$$1\,\mathrm{GB} = 10^9\,\mathrm{B} = 8\times 10^9\,\mathrm{bits}.$$

   Compare this to the total data storage on all the computers in the world, which is roughly

$$3\times 10^9\,\mathrm{GB}.$$

2. Consider a sphere of ordinary matter of surface area A. Suppose this sphere has more entropy than a black hole,

$$S' > S_{\mathrm{BH}} = \frac{A}{A_0}.$$

   Argue that this violates the Second Law. You may assume that as soon as a system of area A reaches the mass M_A of the corresponding black hole, it immediately collapses to form said black hole. *Hint.* How could you force it to collapse?

3. Calculate the optimal information density in a spherical hard drive of radius r.

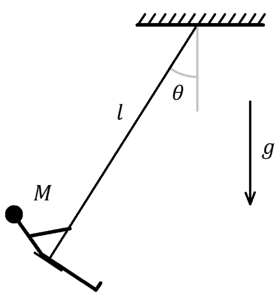
4. Suppose that the speed at which operations can be performed in a hard drive is proportional to the density of information storage. (This is reasonable, since data which is spread out takes more time to bring together for computations.) Explain why huge (spherical) computers are necessarily slow.

[5] At least when it comes to information storage density. *Extracting* information is much harder!

Problem 3 — Swinging on a Swing

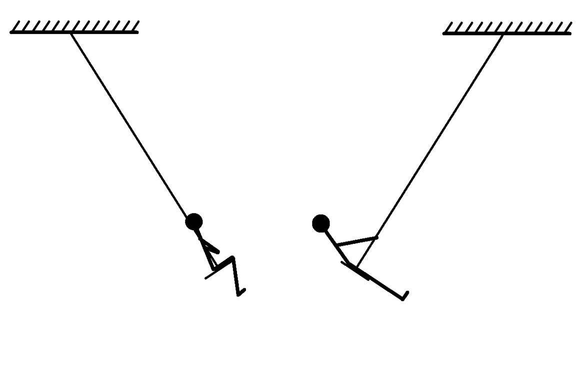
Swinging is a part of everyone's childhood memories. While as kids we learn how to pump a swing based on experience and through trial and error, the physics involved in this mechanism is also intriguing to study. In this problem, we will study two simplified mechanism for pumping a swing, acknowledging that much more complex and accurate models can be obtained with slight modifications.

To start, we will model the swing as a pendulum of length l and let the rider have a mass M and height h. Also let the amplitude of the pendulum, marked by the maximum deviation angle from the vertical, be indicated by θ. In the following two sub-problems, we will work towards finding the amplitude θ_n after n number of half-oscillations (the time-frame between two consequent full stops of the swing).

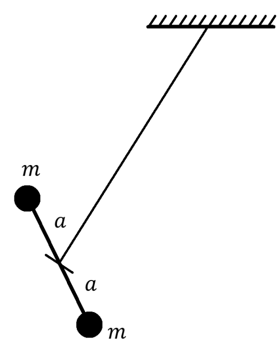


3.1 Pumping from the Sitting Position

This, perhaps more familiar method of pumping, involves the rider switching between the "sat" and the "leaned back" position, animated by the right and the left figure respectively.



This switch, which happens when the swing has come to a full stop at either extreme of its motion, entails a rotation of the rider's body. This rotation in turn causes a small rotation of the swing due to conservation of angular momentum, hence increasing its amplitude. The following is a simplified representation of the rider's body with a barbell of length $2a$ with two masses m attached on either side and attached to the end of the swing at its centre.



1. With reasonable estimates, write a and m in terms of the rider's mass M and height h. Also, what is a reasonable angle of rotation for the barbell to model the rider switching between the "sat" and "leaned back" positions?

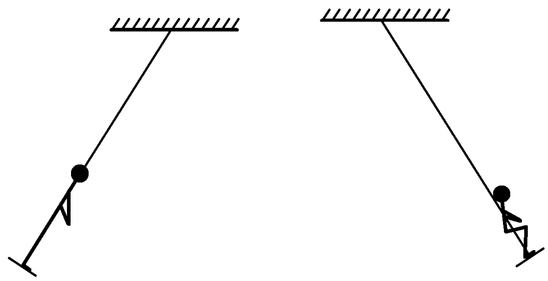
2. At the left-most extreme of swing motion, to increase the amplitude, does the rider switch from the "sat" position to the "leaned back" or vice versa? What about the right-most extreme? Therefore, when should the barbell rotate clockwise and when counter-clockwise?

3. Let us focus on only one switch for now. Write the conservation of angular momentum to obtain the increase in amplitude $\Delta\theta$ in terms of h and l, after the rider performs a switch (Hint: If you have not before, it will help to study the parallel axis theorem $I = I_{cm} + Md^2$).

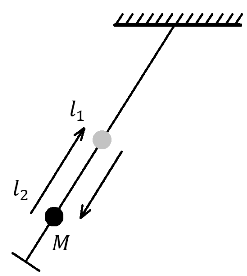
4. Assuming the rider starts pumping the swing from an initial amplitude of θ_0, find the amplitude θ_n after n half-oscillations.

3.2 Pumping from the Standing Position

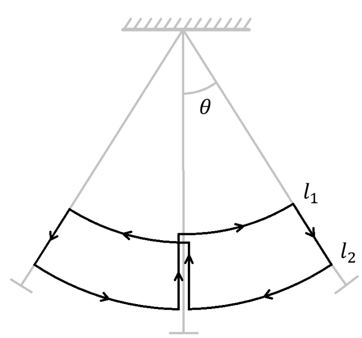
This mechanism of pumping involves the rider switching between the "stood" and "squatted" positions, animated by the left and right figure below respectively.



Switching between these two positions effectively lowers and heightens the rider's centre of mass. We can exploit this ability to pump the swing. First, let us construct a simple representation of the rider with only a mass M that is free to move between two lengths l_1 and l_2 along the length of the swing.



If the rider's centre of mass moves such that it follows the following path, they will be able to pump the swing.



1. With reasonable estimates, write l_1 and l_2 in terms of h and l.

2. Explain why this mechanism will increase the amplitude (Hint: Think about conservation of energy and work done by the rider).

3. Now, assume the swing has an amplitude of θ_n when the rider completes another half-oscillation. Find the new amplitude θ_{n+1} in terms of θ_n, h, and l (Note: You may assume that h/l is sufficiently small to ignore second order and above terms).

4. Assuming the rider starts pumping the swing from an initial amplitude of θ_0, find the amplitude θ_n after n half-oscillations.

5. Compare your answers to part (4) of the two sub-problems. Comment on differences and similarities. Which is a more suitable method of pumping for low amplitudes? What about high amplitudes?