Problem Set No. 4

UBC Metro Vancouver Physics Circle 2018-2019

January 31, 2019

Problem 1 — Hubble's Law and Dark Energy

If we point a telescope at random in the night sky, we discover something surprising: faraway galaxies and stars are all moving away from us.[1] Even more surprising, the speed v of any object is proportional to its distance d from the earth, with

$$v = Hd.$$

The parameter H is called the *Hubble constant* (though it can in fact change), and the relation between velocity and distance is called *Hubble's law*.

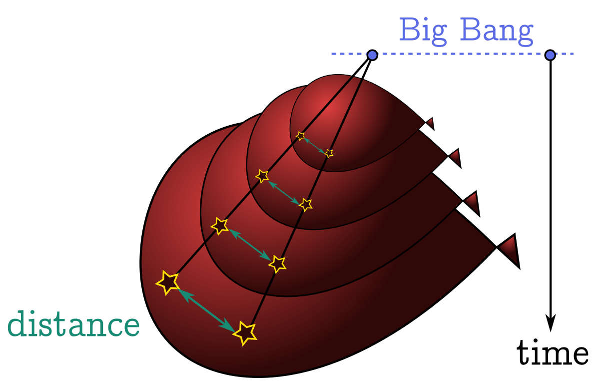


Figure 1: The cosmic balloon, inflated by dark energy.

[1]How? Well, we know what frequencies of light stars like to emit since they are made of chemicals we find on earth. These frequencies are *Doppler-shifted*, or stretched, if the stars in a galaxy are moving away from us, allowing us to determine the speed of recession. Distance is a bit harder to work out, with different methods needed for different distance scales.

A simple analogy helps illustrate. Imagine the universe as a balloon, with objects (like the stars in the image above) in a fixed position on the balloon "skin". Both the distance and relative velocity of any two objects will be proportional to the size of the balloon, and hence each other. The constant of proportionality is H.

1. The universe is expanding. Explain why Hubble's law implies that it does so at an *accelerating* rate.

2. The Virgo cluster is around 55 million light years away and receding at a speed of $1200\,\mathrm{km\,s^{-1}}$. By running time backwards, explain why you expect a Big Bang where everything is located at the same point. From the Virgo cluster and Hubble's law, estimate the age of the universe.

3. Since gravity is an attractive force, the continual expansion of the universe is somewhat mysterious. Why doesn't all the mass collapse back in on itself? The answer to this question is *dark energy.* Although we're not entirely sure what dark energy is, we can model it as an energy density ρ due to empty space itself. This energy does not change with time, since the vacuum always looks the same.

   The state-of-the-art description of gravity is Einstein's theory of *general relativity.* For our purposes, all we need to know is that gravitational effects are governed by Newton's constant G and the speed of light c, where

   $$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}, \quad c = 3 \times 10^8\,\mathrm{m\,s^{-1}}.$$

   Using dimensional analysis, argue that Hubble's constant is related to dark energy by

   $$H^2 = \frac{\eta G \rho}{c^2}$$

   for some (dimensionless) number η. This is the *Friedmann equation.*

4. Assuming that $\eta \sim 1$, estimate the dark energy density of the universe.

Problem 2 — Donuts and Wobbly Orbits

Take a square of unit length. By folding twice and gluing (see below), you can form a donut. Particles confined to the donut don't know it's curved; it looks like normal space to them, except that if they go too far to the left, they will reappear on the right, and similarly for the top and bottom. Put a different way, the blue lines to the left and right are identified, and similarly for the red lines. This is just like the video game *Portal*!

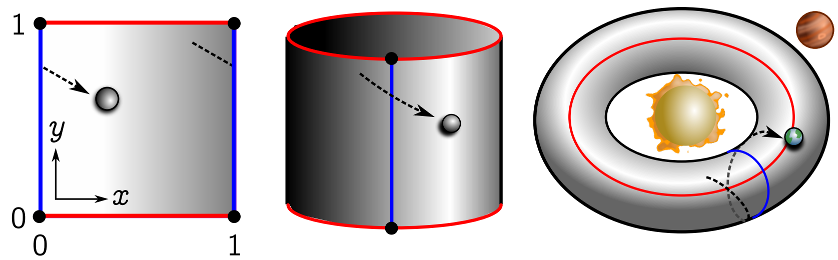


Figure 2: Folding and gluing a square to get a donut. The earth has a wobbly donut orbit (highly exaggerated) due to its attraction to Jupiter.

1. Suppose we have two particles, and shoot them out from the origin at $t = 0$. One particle travels vertically in the y direction with speed v_y, and the other travels in the x direction with speed v_x. Will they ever collide? If so, at what time will the first collision occur?

2. Now consider a *single* particle with velocity vector $\mathbf{v} = (v_x, v_y)$. Show that the particle will never visit the same location on the donut twice if the slope of its path cannot be written as a fraction of whole numbers. Such a non-repeating path is called *non-periodic*.

The earth orbits the sun, but feels a slight attraction to other planets, in particular the gas giant Jupiter. This attraction will deform the circular[2] orbit of the earth onto the surface of a donut, travelling like the particle in question (2). Sometimes, these small changes can accumulate over time until the planet flies off into space! This is obviously something we want to avoid. There is a deep mathematical result[3] which states that the orbit on the donut will be stable provided it is non-periodic. Periodic donut orbits, on the other hand, will reinforce themselves over time and create instabilities. This is like pushing a swing in

[2]In fact, the orbit is slightly stretched along one direction to form an ellipse, but we will ignore this point. One complication at a time!

[3]Called the *KAM theorem* after Kolmogorov, Arnol'd and Moser.

sync with its natural rhythm: eventually, the occupant of the swing will fly off into space as well!

3. Regarding the x-direction as the circular direction around the sun, and y as the direction of the wobbling due to Jupiter, it turns out that

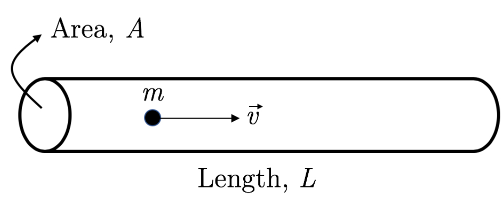
$$\frac{v_y}{v_x} = \frac{T_{\text{Jupiter}}}{T_{\text{earth}}}.$$

If the relative size of orbits is $R_{\text{Jupiter}} = 5R_{\text{Earth}}$, will the earth remain in a stable donut-shaped orbit? *Hint:* You may use the fact that $\sqrt{125}$ cannot be written as a fraction.

Problem 3 — Equation of State

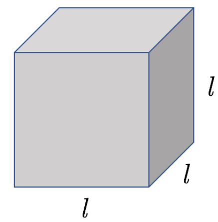
The equation of state relates the physical properties of a system such as volume, pressure, temperature, and density. We will be deriving the equation for an ideal gas, assuming the gas particles are isotropic (properties are the same in all directions) and do not interact with each other — $PV = nRT$.

1. Take a look at a gas particle in a "1D pipe". Find the pressure exerted by one gas particle on the end of the pipe and express it in terms of the particle's mass, velocity, and the volume of the pipe.



2. Now consider N number of gas particles in a cube. Assuming the ideal gas properties and using the result from part 1, derive the equation of state. HINT: Boltzmann constant (k_B) relates the average kinetic energy of the gas particles and the temperature of the gas:

$$\frac{1}{\beta} = k_B T = m{v_x}^2 \text{ where } k_B = \frac{R}{N_A}$$



Avogadro's number: $N_A = 6.022 \times 10^{23}$ particles/mole

Bonus. The relationship given in the hint of part 2 can be proven with a little bit of calculus. Show that this relationship ($^1/_\beta = m{v_x}^2$) is true when the probability of a particle having a specific energy $P = \frac{e^{-\beta E}}{z}$, where $z = \sum e^{-\beta E}$. (Hint: Integrate with respect to v_x, v_y, and v_z.)

Problem 4 — Gone Fishin'

After a day hard at work on kinematics, Emmy decides to take a break from physics and go fishing in nearby Lake Lagrange. But there is no escape! As she prepares to cast her lure, she realizes she has an interesting ballistics problem on her hands.

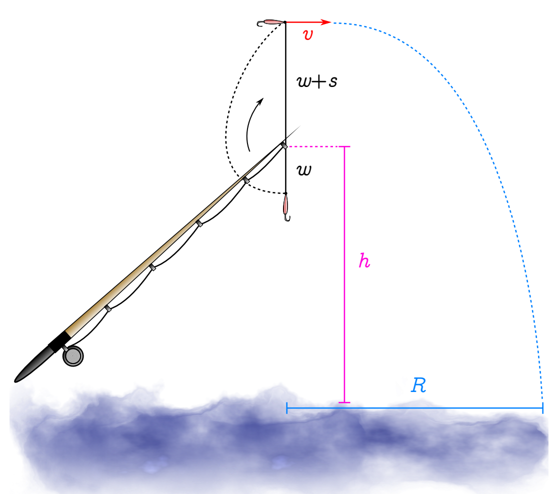


Figure 3: Emmy's unconventional method for casting a lure.

1. The top of her rod is a distance h above the water, and the lure (mass m) hangs on a length of fishing wire w. To cast, Emmy will swing the lure 180° around the end of

the rod and release at the highest point, where the velocity has no vertical component. Assuming she can impart angular momentum $L = mvw$ to the lure, calculate the range R in terms of h, w, L and m. You can ignore the effect of gravity during the initial swinging phase.

2. As Emmy swings through, she can introduce some additional slack s into the wire. Assuming conservation of angular momentum, this will slow the lure but raise the release point. Find the range R in terms of the parameters w, s, h, L, and determine the amount of slack s that maximises the casting distance. Again, ignore the effect of gravity during the swing.

   Hint. Try maximising the *square* of the range.

3. Now include gravity in the swinging phase, and calculate the range as a function of s. Determine the optimum s, and find a condition on h, w, L, g, m which ensures $s > 0$.

   Hint. Complete the square!