Solutions to Problem Set No. 4

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Hubble's Law and Dark Energy

1. Hubble's law says that

$$v = Hd.$$

Assuming that H is constant, the rate of change of the left side is just the acceleration a, while the rate of change of the right side is v, multiplied by the constant H. So

$$a = Hv = H^2 d.$$

Since the universe is expanding, d increases with time. Hence, the acceleration also increases with time!

2. Let's run time backwards until a faraway object collides with us. If the distance is d, and the velocity v, then by Hubble's law the time needed to hit us is

$$t_{\text{collision}} = \frac{d}{v} = \frac{1}{H}.$$

Since this is the same for any object, it suggests that a time $t_{\text{collision}}$, every object in the universe was in the same place. This must be the Big Bang! The age of the universe is then $t_{\text{collision}}$, which we can estimate from the Virgo cluster as

$$t_{\text{collision}} = \frac{d}{v} = \frac{53 \times 10^6 \times (3 \times 10^8\,\text{m/s})}{1.2 \times 10^6\,\text{m/s}}\,\text{years} \approx 13.75 \times 10^9\,\text{years}.$$

We guess the universe is about 13.75 billion years old. The current best estimate is 13.80 billion years!

3. We let L, M, T denote the dimensions of length, mass and time respectively. We know from the previous question that H has the units of inverse time, $[H] = T^{-1}$, and the speed of light clearly has dimensions $[c] = L/T$. We can also find the dimensions of G from the dimensions of the Newton:

$$[\mathrm{N}] = \frac{ML}{T^2} \quad \Longrightarrow \quad [G] = \frac{[\mathrm{N}][\mathrm{m}]^2}{[\mathrm{kg}]^2} = \frac{L^3}{T^2 M}.$$

Finally, since the dimensions of energy are $[E] = ML^2/T^2$, the dimensions of energy *density* (energy over volume) are

$$[\rho] = \frac{[E]}{[V]} = \frac{M}{LT^2}.$$

Let's look for an equation of the form

$$H^\alpha = \eta G^\beta c^\gamma \rho^\delta$$

which has dimensions

$$\frac{1}{T^\alpha} = \eta \left(\frac{L^3}{T^2 M}\right)^\beta \left(\frac{L}{T}\right)^\gamma \left(\frac{M}{LT^2}\right)^\delta = \eta \left(\frac{L^{3\beta+\gamma-\delta} M^{\delta-\beta}}{T^{2\beta+\gamma+2\delta}}\right).$$

This looks hard, but there is no mass or length on the LHS so

$$\delta - \beta = 3\beta + \gamma - \delta = 0 \quad \Longrightarrow \quad 2\beta + \gamma = 0.$$

But then, matching powers of time on both sides,

$$\alpha = 2\beta + \gamma + 2\delta = 2\delta.$$

The simplest way to satisfy all of these constraints is $\beta = \delta = 1$ and $\alpha = -\gamma = 2$. This gives us the Friedmann equation

$$H^2 = \frac{\eta G \rho}{c^2}.$$

4. To find the density of dark energy, we can simply invert the Friedmann equation to

make ρ the subject, and plug in the age of the universe calculated in part (a):

$$\rho \sim \frac{c^2 H^2}{G} = \frac{(3 \times 10^8)^2}{(6.67 \times 10^{-11})(13.75 \times 10^9 \times 365 \times 24 \times 60^2)^2} \frac{\mathrm{J}}{\mathrm{m}^3} \approx 7 \times 10^{-9} \frac{\mathrm{J}}{\mathrm{m}^3}.$$

Doing the full gravity calculation shows that $\eta = 8\pi/3 \sim 10$, so our answer is too large by a factor of approximately 10. Accounting for this, we guess $\rho \sim 10^{-9} \mathrm{J/m}^3$, which matches the current best estimate to within an order of magnitude.[1]

Problem 2 — Donuts and Wobbly Orbits

1. Since the first particle travels on the red line (y-axis) and the second particle travels on the blue line (x-axis), they will only collide if they both return to the origin at the same time. But this means that both must travel an *integer* distance in the same time, so for some natural numbers m_x, m_y, and some time t,

   $$v_x t = m_x, \quad v_y t = m_y.$$

   Dividing one equation by the other, we find that the ratio of velocities must be a fraction:

   $$\frac{v_x}{v_y} = \frac{m_x}{m_y}.$$

   If m_x, m_y have no common denominators, then the first time the particles coincide for $t > 0$ is when $v_x t = m_x$ and $v_y t = m_y$, so $t = v_x/m_x = v_y/m_y$. If the ratio of velocities is not a fraction, they can never collide.

2. This is just the first problem in disguise! The two particles get associated to the x and y coordinates of the single particle. To begin with, suppose the particle starts at the origin at $t = 0$. Let's look for conditions which stop it from returning there. From the first problem, it will never return to the origin as long as v_x/v_y is *irrational*. But there is nothing special about the origin; the same reasoning shows that if the ratio of

[1] In fact, ρ is the *total* energy density of the universe, including things besides dark energy. While dark energy density does not change with time, other forms of energy are diluted as the universe expands; from the Friedmann equation, this means that H changes with time. Indeed, in the past H was very different. However, dark energy constitutes around 70% of the total density, explaining why our estimate here is still reasonably accurate. It also explains why H is approximately constant, at least in the current epoch of expansion.

velocity components is irrational, it will never return to any position it occupies.[2]

3. Kepler's third law states that the radius of an orbit R and the period T (i.e. the length of the year on the planet) are related by

$$T^2 = \alpha R^3$$

for some constant α which is the same for all planets. Thus,

$$\frac{T_{\text{Jupiter}}}{T_{\text{earth}}} = \frac{\sqrt{\alpha} R_{\text{Jupiter}}^{3/2}}{\sqrt{\alpha} R_{\text{earth}}^{3/2}} = 5^{3/2} = \sqrt{125}.$$

Since this cannot be expressed as a fraction, the results of part (2) show that the orbit is non-periodic. This means that the earth should stay in a stable donut orbit forever![3]

Problem 3 — Equation of State

1. If we denote r as the rate of bouncing off the ends, we achieve

$$\text{time} = \frac{2L}{v} = \frac{1}{r}$$

$$\therefore r = \frac{v}{2L}$$

Given that P is pressure and Δp is the change in momentum

$$\begin{aligned} P &= \frac{F}{A} \\ &= \frac{\Delta p}{t} \cdot \frac{1}{A} \\ &= \frac{\Delta p \cdot r}{A} \\ &= \frac{2mvr}{A} \end{aligned}$$

[2] Something even more remarkable happens: the one-dimensional trajectory of the particle manages to fill in most of the the two-dimensional surface of the donut! (It visits everywhere except a miniscule subset of area zero.)

[3] In fact, Jupiter's orbit is only approximately five times larger. But it remains true that a Jupiter year is some irrational number of earth years, which is the key to the stability of the earth's orbit.

$$= \frac{mv^2}{AL}$$

$$= \frac{mv^2}{V}$$

Since $A \times L = V$.

2. Still working in 1D, if we consider N number of particles:

$$P = \sum_{i=1}^{N} \frac{m}{V} v_i^2 = \frac{m}{V} N \cdot \left(\frac{1}{N} \sum_{i=1}^{N} v_i^2 \right)$$

Considering that $\frac{1}{N}\sum_{i=1}^{N} v_i^2$ is the average velocity expression, $\bar{v}$, we achieve

$$P = \frac{Nm \cdot \bar{v}^2}{V}$$

Since velocity is same in all directions in isotropic conditions in 3D:

$$m\bar{v}_x^2 = m\bar{v}_y^2 = m\bar{v}_z^2 = \frac{PV}{N} \qquad \text{(Eqn. 1)}$$

$$\bar{E}_k = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

$$= \frac{3}{2} \cdot \frac{1}{\beta}$$

Where

$$m\bar{v}_x^2 = m\bar{v}_y^2 = m\bar{v}_z^2 = \frac{1}{\beta} \qquad \text{(Eqn. 2)}$$

By combining Eqn. 1 and Eqn. 2, we achieve:

$$\frac{1}{\beta} = \frac{PV}{N} = k_B T = \frac{R}{N_A} T$$

$$PV = \frac{N}{N_A} RT \qquad \left(\text{where } \frac{N}{N_A} = n\right)$$

$$= nRT \quad \longrightarrow \quad \boxed{PV = nRT}$$

Bonus. We know that the average kinetic energy is

$$\bar{E} = \frac{1}{2}m\big(v_x^2 + v_y^2 + v_z^2\big) \qquad \text{(Eqn. 1)}$$

Using the probability expression given the question, we can compute the average energy:

$$\bar{E} = \sum_i P_i \cdot E_i = \int dv_x dv_y dv_z E \frac{e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}}{\int dv_x dv_y dv_z e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}}$$

The energy, E, above can be replaced with the kinetic energy formula and then split into three different integrals.

$$\bar{E} = \frac{1}{2}m \int dv_x dv_y dv_z (v_x^2 + v_y^2 + v_z^2) \frac{e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}}{\int dv_x dv_y dv_z e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}}$$

$$= \frac{1}{2}m \int dv_x dv_y dv_z (v_x^2) \frac{e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}}{\int dv_x dv_y dv_z e^{-\beta\frac{1}{2}m(v_x^2+v_y^2+v_z^2)}} + ...$$

$$= \frac{1}{2}m \int dv_x (v_x^2) \frac{e^{-\beta\frac{1}{2}m(v_x^2)}}{\int dv_x e^{-\beta\frac{1}{2}m(v_x^2)}} \cdot \int dv_y (v_y^2) \frac{e^{-\beta\frac{1}{2}m(v_y^2)}}{\int dv_y e^{-\beta\frac{1}{2}m(v_y^2)}} \cdot \int dv_z (v_z^2) \frac{e^{-\beta\frac{1}{2}m(v_z^2)}}{\int dv_z e^{-\beta\frac{1}{2}m(v_z^2)}} + ...$$

With some simplifying, we get

$$\bar{E} = \frac{3}{2}m \cdot \frac{\int dv (v^2) e^{-\beta\frac{1}{2}m(v^2)}}{\int dv e^{-\beta\frac{1}{2}m(v^2)}} = \frac{3}{2}m\left(\frac{1}{2} \cdot \frac{2}{\beta m}\right) = \frac{3}{2} \cdot \frac{1}{\beta}$$

Relating this to Eqn. 1:

$$\bar{E} = \frac{1}{2}m\big(v_x^2 + v_y^2 + v_z^2\big) = \frac{3}{2} \cdot \frac{1}{\beta}$$

And since $v_x^2 = v_y^2 = v_z^2$,

$$\boxed{mv_x^2 = mv_y^2 = mv_z^2 = \frac{1}{\beta}}$$

Problem 4 — Gone Fishin'

1. Since the lure is released with no vertical velocity, the time it takes to hit the water is

$$h + w = \frac{1}{2}gt^2 \quad \Longrightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The "muzzle" velocity is $v = L/mw$, so the range r of the lure is

$$R = \frac{L}{mw}\sqrt{\frac{2(h+w)}{g}}.$$

2. Our previous answer for range is simply modified by making the replacement $w \to w+s$, but keeping the angular momentum L fixed:

$$R = \frac{L}{m(w+s)}\sqrt{\frac{2(h+w+s)}{g}}.$$

We would like to maximise this distance. We can ignore the constants L, m and $g/2$, write $x = w + s$, and focus on maximising

$$f(x) = \frac{\sqrt{h+x}}{x}.$$

Since this is positive, we can maximise this just as well by maximising its *square* as the hint suggests:

$$F(x) = f^2(x) = \frac{h+x}{x^2}.$$

It's not hard to show that this is a *decreasing* function, so that the best strategy is for Emmy to introduce no slack at all. Let's check that this is true, assuming $0 < x < z$ and trying to show that $F(x) > F(y)$, or even better, $F(x) - F(z) > 0$. We have

$$\begin{aligned} F(x) - F(z) &= \frac{h+x}{x^2} - \frac{h+z}{z^2} \\ &= \frac{(h+x)z^2 - (h+z)x^2}{x^2z^2} \\ &= \frac{h(z^2-x^2) + xz(z-x)}{x^2z^2}. \end{aligned}$$

Since $z > x$, we have $z^2 > x^2$, so the numerator is positive. The denominator is also positive, which means that the whole expression is positive! So the maximum range occurs for $s = 0$.

3. If Emmy adds slack s during the swing, then the lure will undergo a change in height $\Delta y = 2w + s$. This causes the lure to gain gravitational potential energy

$$\Delta U = mg\Delta y = mg(2w + s),$$

leading to a reduced release velocity v':

$$\Delta K = \frac{1}{2}m[(v')^2 - v^2] = -\Delta U \quad \Longrightarrow \quad v' = \sqrt{v^2 - 2g(2w + s)}.$$

Plugging in $v = L/mw$, the range is now

$$R = \sqrt{\frac{2(h + w + s)}{g}\left[\frac{L^2}{m^2w^2} - 2g(2w + s)\right]}.$$

The question now is how to optimise this horrible looking expression! Once again, we can square R, throw away some constants which sit out front, and maximise the very simple function

$$F(s) = (A + s)(B - s),$$

where

$$A = h + w, \quad B = \frac{L^2}{2gm^2w^2} - 2w.$$

By completing the square, we can write

$$F(s) = -\left(s - \tfrac{1}{2}(A - B)\right)^2 + \frac{1}{4}(A - B)^2.$$

Only the first part is relevant to figuring out the optimal s. The function $F(s)$ will be maximised for

$$s = \frac{1}{2}(A - B) = h + 3w - \frac{L^2}{2gm^2w^2}.$$

Of course, for this to be positive, we require $A > B$, or equivalently

$$h + 3w > \frac{L^2}{2gm^2w^2}.$$