Problem Set No. 5

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Turbulence in a Tea Cup

Stir a cup of coffee vigorously enough, and the fluid will begin to mix in a chaotic or *turbulent* way. Unlike the steady flow of water through a pipe, the behaviour of turbulent fluids is unpredictable and poorly understood. However, for many purposes, we can do suprisingly well by modelling a turbulent fluid as a collection of (three-dimensional) eddies of different sizes, with larger eddies feeding into smaller ones and losing energy in the process.

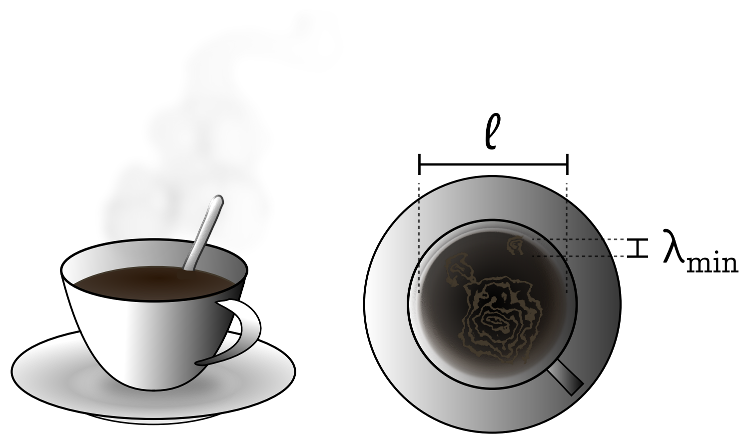

Figure 1: A well-stirred cup of coffee. On the right, a large eddy (size $\sim \ell$) and the smallest eddy (size λ_{min}) are depicted.

Suppose our cup of coffee has characteristic length ℓ, and the coffee has density ρ. When it is turbulently mixed, the largest eddies will be a similar size to the cup, order ℓ, and experience fluctuations in velocity of size Δv due to interaction with other eddies. The fluid also has internal drag[1] or *viscosity* μ, with units $\mathrm{N} \cdot \mathrm{s/m}^2$.

[1]More precisely, viscosity is the resistance to *shear flows*. A simple way to create shear flow is by moving

1. Let ϵ be the rate at which kinetic energy dissipates per unit mass due to eddies. Observation shows that this energy loss is independent of the fluid's viscosity. Argue on dimensional grounds that
$$\epsilon \approx \frac{(\Delta v)^3}{\ell}.$$
Why doesn't the density ρ appear?

2. Kinetic energy can also be lost due to internal friction. Argue that the time scale for this dissipation due to viscosity is
$$\tau_{\text{drag}} \approx \frac{\ell^2 \rho}{\mu}.$$

3. Using the previous two questions, show that eddy losses[2] dominate viscosity losses provided
$$\frac{\ell \rho \Delta v}{\mu} \gg 1.$$
The quantity on the left is called the *Reynolds number*, $\mathrm{Re} = \ell\rho\Delta v/\mu$. In fact, one *definition* of turbulence is fluid flow where the Reynolds number is high.

4. So far, we have focused on the largest eddies. These feed energy into smaller eddies of size λ and velocity uncertainty Δv_λ, which have an associated *eddy Reynolds number*,
$$\mathrm{Re}_\lambda = \frac{\lambda \rho \Delta v_\lambda}{\mu}.$$
When the eddy Reynolds number is less than 1, eddies of the corresponding size are prevented from forming by viscosity.[3] Surprisingly, the rate of energy dissipation per unit mass in these smaller eddies is ϵ, the same as the larger eddies.[4] Show from

a large plate along the surface of a stationary fluid. Experiments show that the friction per unit area of plate is proportional to the speed we move it, and inversely proportional to the height; the proportionality constant at unit height is the viscosity. Since layers of fluid also generate shear flows, viscosity creates internal friction.

[2] Since ϵ depends on $\ell, \Delta v$, you need not consider it when finding the time scale for eddy losses.

[3] Lewis Fry Richardson not only invented the eddy model, but this brilliant mnemonic couplet: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

[4] This is not at all obvious, but roughly, follows because we can fit more small eddies in the container. Intriguingly, this makes the turbulent fluid like a *fractal*: the structure of eddies repeats itself as we zoom in, until viscosity begins to play a role. At infinite Reynolds number, it really is a fractal!

dimensional analysis that the minimum eddy size is roughly

$$\lambda_{\min} \approx \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. If a cup of coffee is stirred violently to Reynolds number $\mathrm{Re} \approx 10^4$, estimate the size of the smallest eddies in the cup.

Problem 2 — Shallow Water makes Tall Waves

Ocean waves behaves rather differently in deep and shallow water. From dimensional analysis, we can learn a little about these differences, including the fact that waves increase in height as they approach the shore. This phenomenon, called *shoaling*, is responsible for tsunamis.

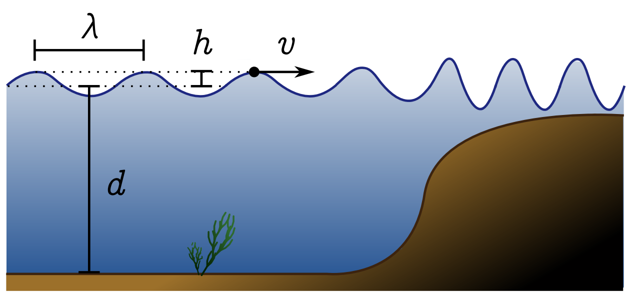

Figure 2: Ocean waves. As the water gets shallower, the waves increase in height.

1. Let λ denote the wavelength of an ocean wave and d the depth of the water. Typically, both are much larger than the height h of the wave, so we can ignore it for the time being. Argue from dimensional analysis that in the *deep water limit* $\lambda \ll d$, the velocity of the wave is proportional to the square root of the wavelength:

$$v \approx \sqrt{g\lambda}.$$

In the *shallow water limit* $\lambda \gg d$, explain why you expect

$$v \approx \sqrt{gd}.$$

2. Ocean waves can be generated by oscillations beneath the ocean floor. For a source of frequency f, what is the wavelength of the corresponding wave in shallow water?

Estimate the wavelength if the source is an earthquake of period $T = 20\,\mathrm{min}$ at depth $d = 4\,\mathrm{km}$, and check your answer is consistent with the shallow water limit.

3. Consider an ocean wave of height h and width w. The energy E carried by a single "cycle" of the wave equals the volume V of water above the mean water level d, multiplied by the gravitational energy density ϵ. By performing a dimensional analysis on each term separately, argue that the total energy in a cycle is approximately

$$E \approx V\epsilon \approx \rho g \lambda w h^2,$$

where $\rho \approx 10^3\,\mathrm{kg\,m^{-3}}$ is the density of water and g the gravitational acceleration.

4. Energy in waves is generally *conserved*: the factor E is constant, even as the wavelength λ and height h of the wave change. (We will ignore spreading of the wave.) By applying energy conservation to shallow waves, deduce *Green's law*:

$$h \propto \frac{1}{d^{1/4}}.$$

The increase in height is called *shoaling*. The relation breaks down near shore when the depth d becomes comparable to the height h.

5. Our earthquake from earlier creates a tsunami of height $h_0 = 0.5\,\mathrm{m}$. What is the height, speed, and power per unit width of the tsunami close to the shore? (By "close to the shore", we mean at $h \approx d$ where Green's law breaks down.) You may assume the shallow water equation holds.[5]

[5] It doesn't quite — we actually need to use the full formula for speed, $v = \sqrt{(g\lambda/2\pi)\tanh(2\pi d/\lambda)^{-1}}$. But it will suffice for an order of magnitude estimate.

Problem 3 — Life of a Sun

The Sun provides energy to our solar system, and maintains life on Earth. Without it, all life would cease to exist and humans would have to migrate. Do you ever wonder how long our Sun will last and how long we have before we need to move to another system? Let's find out!

Figure 3: A beautiful picture of our Sun.

Currently, our Sun is 4.603 billion years old; this is known by radioactive dating. The Sun emits energy at 3.8×10^{26} Watts. This is achieved by fusing hydrogen into helium. ($4H^+ \rightarrow He + \text{energy}$).

1. Using $E = mc^2$, calculate the energy each reaction releases. (M_{H^+}: 1.7×10^{-27} kg, M_{He}: 6.7×10^{-27} kg.)

2. Calculate how many reactions each second is needed for the Sun to continue to output its current energy.

3. Calculate the mass of hydrogen used up every second by the Sun.

4. Suppose the Sun has mass of 2×10^{30} kg, and 10% of that mass can be used for hydrogen fusion. How long is the life of the Sun in years?

5. Given the current state of the Sun, how much mass does it have left to burn?

6. Suppose each generation is about 25.5 years, how many generations do humans have left before they're forced to move to another star system?

Problem 4 — Springy Masses

A mass m is attached to a spring with uncompressed length l_0 and spring constant k as shown below. The spring is fully compressed and then released to propel the mass upward.

1. Find the maximum height h_{max} attained by the mass before returning back to the ground, assuming a sufficiently high spring constant.

2. Find the minimum spring constant k_{min} required for the spring to lose contact with the ground.

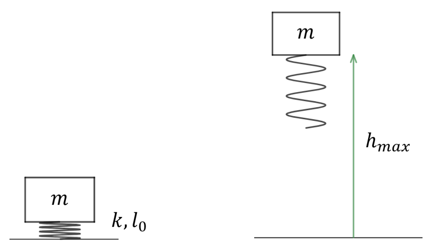

Figure 4: The mass is propelled upwards by the spring.

We will now perform the same experiment with two masses m attached to the same spring. This time, the masses will oscillate while ascending.

3. Find the minimum spring constant k_{min} required for the bottom mass to lose contact with the ground.

4. Assuming a spring constant greater than k_{min}, find the maximum height h_{max} attained by the top mass before returning back to the ground.

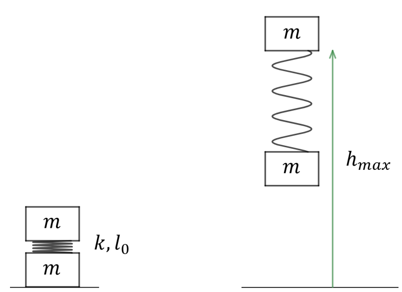

Figure 5: The two masses are propelled upwards while oscillating.