Solutions to Problem Set No. 5

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Turbulence in a Tea Cup

1. Let $[\cdot]$ denote the dimensions of a physical quantity, and M, L, T mass, length and time respectively. Then energy per unit mass per unit time has dimension

$$[\epsilon] = \frac{\text{energy}}{MT} = \frac{M(L/T)^2}{MT} = \frac{L^2}{T^3},$$

where we can remember the dimension for energy using kinetic energy, $K = mv^2/2$. (The dimension does not depend on what form of energy we look at.) The dimensions for the remaining physical quantities are easier:

$$[\ell] = L, \quad [\rho] = \frac{M}{L^3}, \quad [\Delta v] = \frac{L}{T}.$$

Since mass does not appear in $[\epsilon]$, and the viscosity is not involved in this type of dissipation, the density ρ cannot appear since there is nothing besides μ to cancel the mass units. We can easily combine ℓ and Δv to get something with the correct dimension, and deduce an approximate relationship between $\epsilon, \Delta v$ and ℓ:

$$\frac{[(\Delta v)^3]}{[\ell]} = \frac{L^3}{LT^3} = [\epsilon] \quad \Longrightarrow \quad \epsilon \approx \frac{(\Delta v)^3}{\ell}.$$

2. Viscosity has dimensions

$$[\mu] = \frac{[\text{N}][\text{s}]}{[\text{m}^2]} = \frac{MLT}{T^2L^2} = \frac{M}{LT}.$$

We can combine with ρ and ℓ to get something with the dimensions of time; Δv is not involved since friction is independent of the eddies. The unique combination with the right units is

$$\frac{[\ell^2\rho]}{[\mu]} = \frac{L^2 \cdot M \cdot LT}{L^3 \cdot M} = T \quad \Longrightarrow \quad \tau_{\text{drag}} = \frac{\ell^2\rho}{\mu}.$$

3. Returning to eddy losses, its easy to cook up a time scale from the basic physical quantities ℓ and Δv:

$$\tau_{\text{eddy}} \approx \frac{\ell}{\Delta v}.$$

In order for dissipation of energy by the eddies to dominate, we require $\tau_{\text{eddy}} \ll \tau_{\text{drag}}$, that is, energy is much more quickly dissipated by the eddies than by friction. Comparing the two expressions, we find

$$\frac{\ell}{\Delta v} \ll \frac{\ell^2\rho}{\mu} \quad \Longrightarrow \quad \frac{\ell\rho\Delta v}{\mu} = \text{Re} \gg 1.$$

4. By assumption, the rate of energy dissipation ϵ is the same for all eddies, so the reasoning in part (1) gives $\epsilon \approx (\Delta v_\lambda)^3/\lambda$. Rearranging, we have $\Delta v_\lambda \approx (\epsilon\lambda)^{1/3}$. We now set $\text{Re}_\lambda = 1$ and solve for the minimum eddy size λ_{min}:

$$1 = \text{Re}_\lambda = \frac{\lambda\rho\Delta v_\lambda}{\mu} \approx \frac{\lambda^{4/3}\epsilon^{1/3}\rho}{\mu} \quad \Longrightarrow \quad \lambda_{\text{min}} \approx \left(\frac{\mu}{\epsilon^{1/3}\rho}\right)^{3/4} = \left(\frac{\mu^3}{\epsilon\rho^3}\right)^{1/4}.$$

5. There is a cute shortcut here. First, the previous question tells us how Re_λ scales with λ:

$$\text{Re}_\lambda \approx \frac{\epsilon^{1/3}\rho\lambda^{4/3}}{\mu} = \alpha\lambda^{4/3},$$

where α is a constant independent of λ. But the Reynolds number is simply the eddy Reynolds number for $\lambda = \ell$, $\text{Re} = \text{Re}_\ell$, and the eddy Reynolds number is unity for the smallest eddies. Hence,

$$\text{Re}_{\lambda_{\text{min}}} \approx \alpha\lambda_{\text{min}}^{4/3} = 1, \quad \text{Re} \approx \alpha\ell^{4/3} \quad \Longrightarrow \quad \lambda_{\text{min}}^{4/3} \approx \frac{\ell^{4/3}}{\text{Re}}.$$

For our turbulent coffee, $\ell \approx 10\,\text{cm}$ and $\text{Re} \approx 10^4$, so we estimate a minimum eddy size

$$\lambda_{\text{min}} \approx \frac{\ell}{\text{Re}^{3/4}} \approx \frac{10\,\text{cm}}{10^3} = 0.1\,\text{mm}.$$

Problem 2 — Shallow Water makes Tall Waves

1. Let's write the dimensions of g and ρ in terms M, L, T:

$$[g] = \frac{L}{T^2}, \quad [\rho] = \frac{M}{L^3}.$$

In deep water $d \gg \lambda$, the wave cannot "see" the bottom of the ocean; it is too far away. We only expect the smaller length λ to control the speed. To find the velocity v with dimensions $[v] = L/T$, we can combine g, ρ and λ in precisely one way:

$$v = \sqrt{g\lambda}.$$

It turns out that ρ is not involved! There is no other term to cancel the dimension of mass. Similarly, in shallow water $d \ll \lambda$, the depth is more important than the wavelength, so that we instead get

$$v = \sqrt{gd}.$$

2. The velocity is related to the wavelength and frequency by $v = f\lambda$. Hence, the wavelength of a wave in shallow water of depth d is fixed by question (2):

$$\lambda = \frac{v}{f} = \frac{\sqrt{gd}}{f}.$$

Let's plug in the numbers for the earthquake, noting that $f = 1/T$:

$$\lambda = \sqrt{9.8 \cdot 4000} \cdot (20 \cdot 60)\,\mathrm{m} \approx 237\,\mathrm{km}.$$

This is much larger than than the depth of the ocean, so we can consistently use the shallow water limit.

3. For simplicity, we treat one cycle of the wave as a box, whose volume is the product of length, width and height:

$$V \approx hw\lambda.$$

If E is the dimension of energy, then ϵ has dimensions $[\epsilon] = E/L^3$. Since the energy is due to the gravitational potential of the portion of water above the mean water level, it will involve the height h, the density ρ, and the gravitational acceleration g. The gravitational potential energy is mgh, so the energy *density* should be

$$\epsilon \sim \rho g h.$$

We can get the same answer from dimensional analysis, since

$$[\epsilon] = \frac{E}{L^3} = \frac{ML^2}{L^3T^2} = \frac{M}{L^3} \cdot \frac{L}{T^2} \cdot L = [\rho g h],$$

where we used the fact that $E = ML^2T^{-2}$ (using the formula for kinetic energy, for example). Thus, the energy carried by one cycle of the wave is

$$E \approx V\epsilon \approx \rho g \lambda w h^2.$$

4. In shallow waves, question (3) shows that $\lambda \propto \sqrt{d}$. Since ρ, g, w are constant, we have

$$E \propto \sqrt{d} h^2.$$

Taking the square root, and using the fact that E is constant, we obtain Green's law:

$$hd^{1/4} \propto \sqrt{E} \quad \Longrightarrow \quad h \propto \frac{1}{d^{1/4}}.$$

5. The wave is "close to shore" when the height is comparable to the depth of the water, $h \approx d$. We can use this, along with Green's law and the initial height and depth, to determine h:

$$hd^{1/4} = h^{5/4} = h_0 d_0^{1/4} \quad \Longrightarrow \quad h = h_0^{4/5} d_0^{1/5} = 4000^{1/5} \approx 5.25\,\text{m}.$$

Assuming the shallow water equation holds,

$$v \approx \sqrt{gd} \approx \sqrt{9.8 \cdot 5.25}\,\text{m} \cdot \text{s}^{-1} = 7.2\,\text{m} \cdot \text{s}^{-1}.$$

The tsunami is around 5 meters high and travelling at a velocity of $7\,\text{m} \cdot \text{s}^{-1}$. This doesn't sound that high or fast, but is more than enough to cause catastrophic damage.

To see how much energy such a tsunami delivers, we use our expression from part (3). To find the total power, we divide the energy delivered per wave E by the period of the wave, $T = 20\,\text{min}$. To find the power P per unit width, we divide by w. The result is

$$P = \frac{E}{Tw} \approx \frac{\rho g \lambda h^2}{T} = \rho g v h^2,$$

using $v = \lambda/T$. To evaluate this, we plug in the value for v we calculate, and the density of water $\rho \approx 10^3\,\mathrm{kg}\cdot\mathrm{m}^{-3}$. This gives

$$P \approx 10^3 \cdot 9.8 \cdot 7.2 \cdot 5.25^2\,\mathrm{W}\cdot\mathrm{m}^{-1} \approx 2\,\mathrm{MW}\cdot\mathrm{m}^{-1}.$$

The tsunami delivers around 2 megawatts per meter of shoreline. This is enough power for roughly 400 households! Since the tsunami is supplying this amount *for each metre of shoreline*, it's not too hard to see why a tsunami of modest height can still wreak terrible havoc.

Problem 3 — Life of a Sun

1. Using conservation of energy $E_i = E_f$. $E_i = 4(1.7 \times 10^{-27})c^2$ and $E_f = 6.7 \times 10^{-27}c^2 + E_{\text{released}}$. Solving for energy released, we arrive at $E_{\text{released}} = 9 \times 10^{-12}$ J.

2. For each reaction, there is 9×10^{-12} J released. A Sun is known to to have an output power of 3.8×10^{26} W (or J/s). To find the number of reactions:

$$\frac{3.8 \times 10^{26} \text{ J/s}}{9 \times 10^{-12} \text{ J/reaction}} = 4.2 \times 10^{37} \text{ reactions/s}$$

3. Each reaction uses $(4 \cdot 1.7) \times 10^{-27}$ kg of hydrogen. Total mass of hydrogen per second:

$$\frac{(4 \cdot 1.7) \times 10^{-27} \text{ kg}}{\text{reaction}} \times \frac{4.2 \times 10^{37} \text{ reaction}}{\text{s}} = 2.87 \times 10^{11} \text{ kg/s}$$

4. 10% of 2×10^{30} kg is 2×10^{29} kg. To find the life of the Sun:

$$\frac{2 \times 10^{29} \text{ kg}}{2.87 \times 10^{11} \frac{\text{kg}}{\text{s}}} = 6.97 \times 10^{17} \text{ s} = 20 \text{ billion years}$$

Note: The life of the Sun is actually 10 billion years, so why is our calculation off? This is because not enough significant digits were used for part (1); the problem was simplified to focus on dimensional analysis rather than calculation. For those interested: Try to do the calculations again with M_{H^+}: 1.6725×10^{-27} kg, M_{He}: 6.644×10^{-27} kg. and you should find the life of the Sun is indeed 10 billion years. We will continue to use 20 billion years as our answer for further calculations.

5. The Sun is 4.603 billion years old. If we use 20 billion years, it has 15.397 billion years left $= 4.85559792 \times 10^{17}$ s. The Sun burns 2.87×10^{11} kg/s. This means that it still has

$$\frac{2.87 \times 10^{11}\ \mathrm{kg}}{\mathrm{s}} \times\ 4.85559792 \times 10^{17}\ \mathrm{s} = 1.39 \times 10^{29}\ \mathrm{kg\ left\ to\ burn}$$

6. The Sun has 15.397 billion years left. This means we still have

$$\frac{15.397 \times 10^{9}\ \mathrm{years}}{25.5\ \mathrm{years}} = 603{,}803{,}922\ \mathrm{generations\ left}$$

In reality we only have about 211,647,058 generations left because the life of the Sun is actually 10 billion years. However, this is still a very large number!

Problem 4 — Springy Masses

1. Simply writing energy conservation,

$$\frac{k\,{l_0}^2}{2} = mgh_{\max} \implies h_{\max} = \frac{k\,{l_0}^2}{2mg}$$

2. For the spring to lose contact we require $h_{\max} > l_0$; therefore,

$$\frac{k\,{l_0}^2}{2mg} > l_0 \implies k_{\min} = \frac{2mg}{l_0}$$

3. For the bottom mass to lose contact we require the spring force to be $F_s = mg = k\Delta x$. Therefore, the spring extension is $\Delta x = mg/k$. Writing energy conservation again,

$$\frac{k\,{l_0}^2}{2} = mg\left(l_0 + \frac{mg}{k}\right) + \frac{k\left(\frac{mg}{k}\right)^2}{2}$$

$$k^2 - \frac{2mg}{l_0}\cdot k - \frac{3m^2g^2}{{l_0}^2} = 0 \implies k = \frac{3mg}{l_0}$$

4. The motion of the system past loss of contact can be described as the sum of two independent motions: the free fall motion of the center of mass, and the simple harmonic oscillatory motion of the masses with respect to the center of mass.

   Let us first parametrize the center of mass motion. At the point of loss of contact, the velocity of upper mass is given by energy conservation,

$$\frac{k\,{l_0}^2}{2} = mg\left(l_0 + \frac{mg}{k}\right) + \frac{k\left(\frac{mg}{k}\right)^2}{2} + \frac{m{v_0}^2}{2}$$

$$v_0 = \sqrt{\frac{k}{m}\left({l_0}^2 - \left(\frac{mg}{k}\right)^2\right) - 2g\left(l_0 + \frac{mg}{k}\right)} = \sqrt{\frac{k}{m}\left(l_0 + \frac{mg}{k}\right)\left(l_0 - \frac{3mg}{k}\right)}$$

   And hence the motion of center of mass is described by,

$$v_{cm_i} = \frac{mv_0}{2m} = \frac{v_0}{2}$$

$$y_{cm} = \frac{l_0 + \frac{mg}{k}}{2} + \frac{v_0 t}{2} - \frac{gt^2}{2}$$

   And now we will parametrize the oscillatory motion of the top mass with respect to the center of mass. Since the effective spring constant is $k_{eff} = 2k$ (only the top half of the spring is acting on the top mass), the angular frequency of oscillations is $\omega = \sqrt{2k/m}$. The general solution for simple harmonic oscillation and its time derivative are given by,

$$h_1 - y_{cm} = y_m = c_1 \sin(\omega t) + c_2 \cos(\omega t) + \frac{l_0}{2}$$
$$\dot{y}_m = c_1 \omega \cos(\omega t) - c_2 \omega \sin(\omega t)$$

   And c_1, c_2 are determined by initial conditions,

$$y_m(0) = \frac{l_0 + \frac{mg}{k}}{2} = \frac{l_0}{2} + c_2$$
$$\dot{y}_m(0) = \frac{v_0}{2} = c_1 \omega$$

   Therefore, for the height of top mass as a function of time we have,

$$h_1(t) = y_{cm} + y_m = l_0 + \frac{mg}{2k} + \frac{v_0 t}{2} - \frac{gt^2}{2} + \frac{v_0}{2\omega} \cdot \sin(\omega t) + \frac{mg}{2k} \cdot \cos(\omega t)$$

$$v_0 = \sqrt{\frac{k}{m}\left(l_0 + \frac{mg}{k}\right)\left(l_0 - \frac{3mg}{k}\right)}$$

Maximizing this function is not trivial, and in fact cannot be done analytically! Here, we will find the maximum attained height for a specific case by plotting height versus time (you can play with the Desmos graph here: bit.ly/2EvbPKb).

For this graph, we have chosen $m = 50\,\text{g}$, $k = 100\,\text{N/m}$, $g = 9.8\,\text{m/s}^2$, and $l_0 = 10\,\text{cm}$.

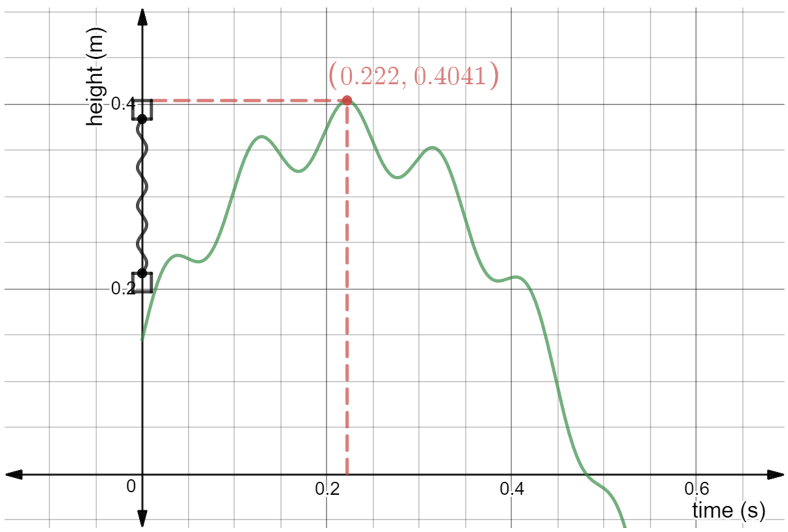

Figure 1: Graph of height of top mass as a function of time. The height function is an overlay of a parabolic and a sinusoidal function.

As you can see, in this case, a maximum height of $h_{\text{max}} = 0.40\,\text{m}$ is reached at $t = 0.22\,\text{s}$.