Solutions to Problem Set No. 7

UBC Metro Vancouver Physics Circle 2018-2019

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Problem 1 — Intense Fireworks

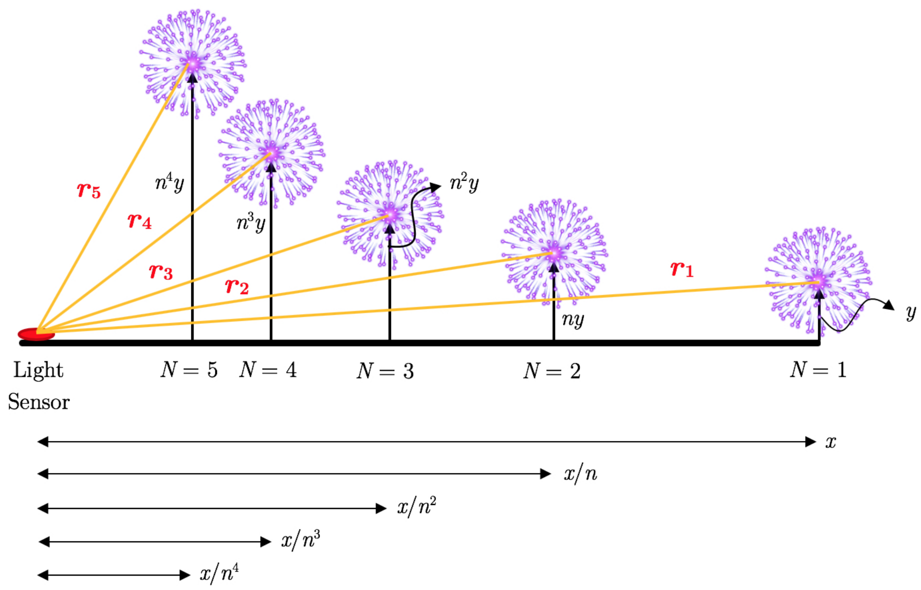
Intensity is the amount of energy a source conveys per unit time across a surface of unit area.

$$I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

Thus, intensity follows an inverse-square law (assuming a source generating constant power),

$$I \propto \frac{1}{r^2}$$

where intensity decreases significantly as the energy is dispersed in space. When we refer to the diagram below,



each firework explosion results in a right angle triangle, where the path of light to the light sensor is the hypotenuse of the triangle. For example, the intensity of the first shot ($N = 1$) can be described as

$$I_1 \propto \frac{1}{(r_1)^2} = \frac{1}{x^2 + y^2}$$

In order to show that $(I_{N+1}/I_N) = 1/n^2$ as N increases indefinitely, we must come up with a general expression of this ratio in terms of N only. An important note to take into account is that when we are taking the ratios, all constants are cancelled out since the only variable changing is r. Let's begin with calculating some r values for different shots.

$$(r_1)^2 = x^2 + y^2$$

$$\begin{aligned}(r_2)^2 &= \left(\frac{x}{n}\right)^2 + (ny)^2 \\ &= \frac{x^2}{n^2} + n^2y^2 \\ &= \frac{x^2 + n^4y^2}{n^2}\end{aligned}$$

$$\begin{aligned}(r_3)^2 &= \left(\frac{x}{n^2}\right)^2 + (n^2y)^2 \\ &= \frac{x^2}{n^4} + n^4y^2 \\ &= \frac{x^2 + n^8y^2}{n^4}\end{aligned}$$

$$\begin{aligned}(r_4)^2 &= \left(\frac{x}{n^3}\right)^2 + (n^3y)^2 \\ &= \frac{x^2}{n^6} + n^6y^2 \\ &= \frac{x^2 + n^{12}y^2}{n^6}\end{aligned}$$

Therefore, based on the pattern that we see above,

$$
\begin{aligned}
(r_N)^2 &= \left(\frac{x}{n^{N-1}}\right)^2 + (n^{N-1}y)^2 \\
&= \frac{x^2}{n^{2N-2}} + n^{2N-2}y^2 \\
&= \frac{x^2 + n^{4N-4}y^2}{n^{2N-2}}
\end{aligned}
$$

Now that we have the r_N expression for any number of shot N, we can determine our consecutive intensity ratio:

$$
\begin{aligned}
\frac{I_{N+1}}{I_N} &= \frac{(r_N)^2}{(r_{N+1})^2} \\
&= \frac{\left(x^2 + n^{4N-4}y^2\right) / \left(n^{2N-2}\right)}{\left(x^2 + n^{4(N+1)-4}y^2\right) / \left(n^{2(N+1)-2}\right)} \\
&= \frac{x^2 + n^{4N-4}y^2}{n^{2N-2}} \cdot \frac{n^{2N}}{x^2 + n^{4N}y^2} \\
&= n^{2N-2N+2} \cdot \frac{x^2 + n^{4N-4}y^2}{x^2 + n^{4N}y^2} \\
&= n^2 \cdot \frac{x^2 + n^{4N-4}y^2}{x^2 + n^{4N}y^2}
\end{aligned}
$$

If we apply the limit as N increases indefinitely, we obtain:

$$
\begin{aligned}
\lim_{N\to\infty} \frac{I_{N+1}}{I_N} &= n^2 \lim_{N\to\infty} \left(\frac{x^2 + n^{4N-4}y^2}{x^2 + n^{4N}y^2}\right) \\
&= n^2 \lim_{N\to\infty} \left(\frac{n^{4N-4}y^2}{n^{4N}y^2}\right) \\
&= n^2 \lim_{N\to\infty} \left(n^{4N-4-4N}\right) \\
&= n^2 \cdot n^{-4}
\end{aligned}
$$

$$
\boxed{\lim_{N\to\infty} \frac{I_{N+1}}{I_N} = \frac{1}{n^2}}
$$

Problem 2 — A Sticky Situation

1. In order to find the expression for $\frac{x}{D}$, our goal is to find x (as D can be easily found later). This quantity is the horizontal range of the combined masses from the half-way point. Since this is within the horizontal direction, we can define it to be

   $$x \equiv \text{(horizontal velocity of combined masses)} \times \text{(time it takes to hit the ground)}$$
   $$x = u_x \tau$$

   Let's begin with the first unknown, u_x, defined as the horizontal velocity of the combined masses. To find this, we must use conservation of momentum. First, let's find an expression for h then use that to find the velocity of impact of M as it drops a height of h. In the vertical direction:

   $$v^2 = v_0^2 + 2ad$$
   $$(v_{m,f})^2 = (v_{m,i})^2 - 2gh$$
   $$0 = (v \sin\theta)^2 - 2gh$$
   $$h = \frac{(v \sin\theta)^2}{2g} \qquad \text{(Eqn. 1)}$$

   $$v^2 = v_0^2 + 2ad$$
   $$(v_{M,f})^2 = (v_{M,i})^2 + 2gh$$
   $$(v_{M,f})^2 = 2g\left(\frac{(v \sin\theta)^2}{2g}\right) \qquad \text{(Using Eqn. 1)}$$
   $$v_{M,f} = v \sin\theta$$

   This essentially means that the final vertical velocity of M is the initial vertical velocity of m, which makes sense as they both traverse the same displacement.

It is now time to use conservation of momentum within the $x-$direction:

$$\begin{aligned} p_i &= p_f \\ p_{m,i} + \cancelto{0}{p_{M,i}} &= p_f \\ mv\cos\theta &= (m+M)u_x \\ \frac{mv\cos\theta}{m+M} &= u_x \end{aligned}$$

If $\alpha = \dfrac{m}{M}$, then our horizontal velocity will be:

$$u_x = \frac{\alpha}{\alpha+1}\, v\cos\theta \qquad \text{(Eqn. 2)}$$

Now, let's use conservation of momentum in the $y-$direction to find the vertical velocity of the combined masses.

$$\begin{aligned} p_i &= p_f \\ \cancelto{0}{p_{m,i}} + p_{M,i} &= p_f \\ Mv\sin\theta &= (m+M)u_y \\ \frac{Mv\sin\theta}{m+M} &= u_y \end{aligned}$$

Again, if $\alpha = \dfrac{m}{M}$, then our vertical velocity will be:

$$u_y = \frac{v\sin\theta}{\alpha+1} \qquad \text{(Eqn. 3)}$$

Let's use this information to calculate τ, the time it takes the combined masses to hit the ground.

$$\begin{aligned} d &= v_0 t + \frac{1}{2}at^2 \\ h &= u_y\tau + \frac{1}{2}g\tau^2 \\ 0 &= \frac{1}{2}g\tau^2 + \frac{v\sin\theta}{\alpha+1}\,\tau - \frac{(v\sin\theta)^2}{2g} \qquad \text{(Using Eqns. 1 and 3)} \end{aligned}$$

To solve for τ, we must use the quadratic equation.

$$\tau = \frac{-\frac{v\sin\theta}{\alpha+1} \pm \sqrt{\frac{(v\sin\theta)^2}{(\alpha+1)^2} - 4\left(\frac{1}{2}g\right)\left(-\frac{(v\sin\theta)^2}{2g}\right)}}{2\left(\frac{1}{2}g\right)}$$

$$= \frac{-\frac{v\sin\theta}{\alpha+1} + \sqrt{\frac{(v\sin\theta)^2}{(\alpha+1)^2} + (v\sin\theta)^2}}{g} \qquad \text{(Choosing only the } \oplus \text{ root.)}$$

$$= \frac{v\sin\theta}{g}\left(-\frac{1}{\alpha+1} + \sqrt{\frac{1}{(\alpha+1)^2} + 1}\right) \qquad \text{(Factoring } v\sin\theta \text{ out.)}$$

$$= \frac{v\sin\theta}{g}\left(-\frac{1}{\alpha+1} + \sqrt{\frac{1+(\alpha+1)^2}{(\alpha+1)^2}}\right)$$

By combining everything together, we achieve:

$$\tau = \frac{v\sin\theta}{g}\left(\frac{\sqrt{(\alpha+1)^2+1}-1}{\alpha+1}\right) \qquad \text{(Eqn. 4)}$$

Now that we've calculated for both u_x (Eqn. 2) and τ (Eqn. 4), we can simply multiply the two together to determine the horizontal displacement of the combined masses:

$$x = \frac{v^2\sin\theta\cos\theta}{g} \cdot \frac{\alpha\left(\sqrt{(\alpha+1)^2+1}-1\right)}{(\alpha+1)^2} \qquad \text{(Eqn. 5)}$$

In the original diagram, D is half of the range, R. To calculate R:

$$d = v_0 t + \frac{1}{2}at^2 \qquad R = (v\cos\theta)t$$

$$0 = (v\sin\theta)t - \frac{1}{2}gt^2 \qquad R = \frac{2v^2\sin\theta\cos\theta}{g}$$

$$t = \frac{2v\sin\theta}{g} \qquad \therefore D = \frac{v^2\sin\theta\cos\theta}{g} \qquad \text{(Eqn. 6)}$$

Finally, we obtain what we are looking for when we divide Eqn. 5 by Eqn. 6:

$$\boxed{\frac{x}{D} = \frac{\alpha\left(\sqrt{(\alpha+1)^2+1}-1\right)}{(\alpha+1)^2}}$$

2. If $f(\alpha) = \dfrac{\alpha\left(\sqrt{(\alpha+1)^2+1}-1\right)}{(\alpha+1)^2}$, then:

$$\begin{aligned}
\lim_{\alpha\to\infty} f(\alpha) &= \lim_{\alpha\to\infty} \frac{\alpha\left(\sqrt{(\alpha+1)^2+1}-1\right)}{(\alpha+1)^2} \\
&= \lim_{\alpha\to\infty} \frac{\alpha\left(\sqrt{\alpha^2+2\alpha+2}-1\right)}{\alpha^2+2\alpha+1} \\
&= \lim_{\alpha\to\infty} \frac{\alpha\left(\alpha\sqrt{1+\frac{2}{\alpha}+\frac{2}{\alpha^2}}-1\right)}{\alpha^2\left(1+\frac{2}{\alpha}+\frac{1}{\alpha^2}\right)} \\
&= \lim_{\alpha\to\infty} \frac{\alpha^2\left(\sqrt{1+\frac{2}{\alpha}+\frac{2}{\alpha^2}}-\frac{1}{\alpha}\right)}{\alpha^2\left(1+\frac{2}{\alpha}+\frac{1}{\alpha^2}\right)} \\
&= \lim_{\alpha\to\infty} \frac{\sqrt{1+\frac{2}{\alpha}+\frac{2}{\alpha^2}}-\frac{1}{\alpha}}{1+\frac{2}{\alpha}+\frac{1}{\alpha^2}} \\
&= \lim_{\alpha\to\infty} \frac{\sqrt{1+\cancelto{0}{\frac{2}{\alpha}}+\cancelto{0}{\frac{2}{\alpha^2}}}-\cancelto{0}{\frac{1}{\alpha}}}{1+\cancelto{0}{\frac{2}{\alpha}}+\cancelto{0}{\frac{1}{\alpha^2}}} \\
&= \frac{\sqrt{1}}{1} = 1
\end{aligned}$$

$$\boxed{\therefore \lim_{\alpha\to\infty} f(\alpha) = 1}$$

3.

$$
\begin{aligned}
\frac{\alpha\left(\sqrt{(\alpha+1)^2+1}-1\right)}{(\alpha+1)^2} &= \frac{1}{2} \\
\sqrt{(\alpha+1)^2+1} &= \frac{(\alpha+1)^2}{2\alpha}+1 \\
(\alpha+1)^2+1 &= \left(\frac{(\alpha+1)^2}{2\alpha}+1\right)^2 \\
\alpha^2+2\alpha+2 &= \frac{\left((\alpha+1)^2+2\alpha\right)^2}{4\alpha^2} \\
4\alpha^4+8\alpha^3+8\alpha^2 &= (\alpha^2+4\alpha+1)^2 \\
4\alpha^4+8\alpha^3+8\alpha^2 &= \alpha^4+8\alpha^3+18\alpha^2+8\alpha+1 \\
3\alpha^4-10\alpha^2-8\alpha-1 &= 0
\end{aligned}
$$

By the zero factor principle, $\alpha = -1$ is a potential solution:

$$3(-1)^4 - 10(-1)^2 - 8(-1) - 1 = 3 - 10 + 8 - 1 = 0$$

Knowing this, we can use long division to factor the polynomial.

$$
\begin{array}{r}
3\alpha^3 - 3\alpha^2 - 7\alpha - 1 \\
\alpha+1\,\big)\,\overline{3\alpha^4 + 0\alpha^3 - 10\alpha^2 - 8\alpha - 1} \\
-\ \underline{3\alpha^4 + 3\alpha^3} \qquad\qquad\qquad\quad \\
-3\alpha^3 - 10\alpha^2 \qquad\quad \\
-\ \underline{-3\alpha^3 - 3\alpha^2} \qquad\quad \\
-7\alpha^2 - 8\alpha \quad \\
-\ \underline{-7\alpha^2 - 7\alpha} \quad \\
-\alpha - 1 \\
-\ \underline{-\alpha - 1} \\
0
\end{array}
$$

By the same principle, $\alpha = -1$ is yet again another potential solution, thus $\alpha + 1$ is another factor:

$$3(-1)^3 - 3(-1)^2 - 7(-1) - 1 = -3 - 3 + 7 - 1 = 0$$

If we use long division again,

$$(3\alpha^3 - 3\alpha^2 - 7\alpha - 1) \div (\alpha + 1) = 3\alpha^2 - 6\alpha - 1$$

Therefore, when we factor the polynomial we get:

$$3\alpha^4 - 10\alpha^2 - 8\alpha - 1 = (\alpha + 1)^2(3\alpha^2 - 6\alpha - 1)$$

If we use the the quadratic equation to solve the last factor, we get:

$$\begin{aligned}
\alpha &= \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} \\
&= \frac{6 \pm \sqrt{36 + 12}}{6} \\
&= \frac{6 \pm \sqrt{48}}{6} \\
&= \frac{6 \pm 4\sqrt{3}}{6} \\
&= \frac{3 \pm 2\sqrt{3}}{3}
\end{aligned}$$

Therefore, our potential solutions are

$$\alpha = -1, \frac{3 \pm 2\sqrt{3}}{3}$$

However, $\alpha = -1$ or $\alpha = \frac{3 - 2\sqrt{3}}{3}$ are neither mathematically true nor physically possible. Therefore, to get exactly half the half-range, or $f(\alpha) = \frac{1}{2}$,

$$\boxed{\alpha = \frac{3 + 2\sqrt{3}}{3} \approx 2.2 \text{ times more massive than } M}$$