Problem Set No. 10

UBC Metro Vancouver Physics Circle 2018-2019

May 16, 2019

1 Einstein Rings

Problem. According to *general relativity*, Einstein's theory of gravity, massive objects curve space itself. Even *massless* particles like light rays will be deflected as they try to find the shortest path between A and B. This effect is called *gravitational lensing*, since a heavy body acts like a lens.

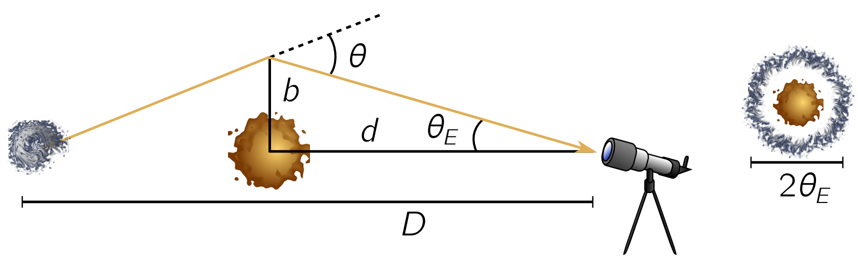


Figure 1: Einstein ring, gravitationally lensed by a large star.

1. Suppose that a light ray passes a spherical star of mass m and radius R, a distance $b > R$ from the centre. The *angle of deflection* θ is dimensionless (in radians). Using dimensional analysis, argue that it takes the form

$$\theta = c_0 + c_1 x + c_2 x^2 + \cdots$$

for dimensionless constants $c_0, c_1, c_2, \ldots$, and

$$x = \frac{Gm}{bc^2}.$$

The speed of light is $c = 3\times10^8$ m/s and Newton's constant is $G = 6.7\times10^{-11}$ m^3/kg s^2.

Hint. You may assume the wavelength of light isn't relevant, since only the mass $m = 0$ determines its path in spacetime. Why isn't the radius of the star relevant?

2. By considering the limit where the star disappears altogether, $m \to 0$, explain why $c_0 = 0$.

3. Using parts (1) and (2), argue that for $Gm \ll bc^2$,

$$\theta \sim \frac{Gm}{bc^2}.$$

As usual, the $\sim$ includes the unknown constant c_1.

Imagine that a star lies directly between a galaxy and a telescope on earth.The galaxy is a distance D away from the earth, and the star a distance d. Define the angle θ_E and deflection angle θ as in Fig. 1.

4. Assuming the angles are small, argue that

$$b \approx \theta_E d, \quad \theta_E D \approx \theta(D - d).$$

5. Combining the identities in (4) with (3), deduce that

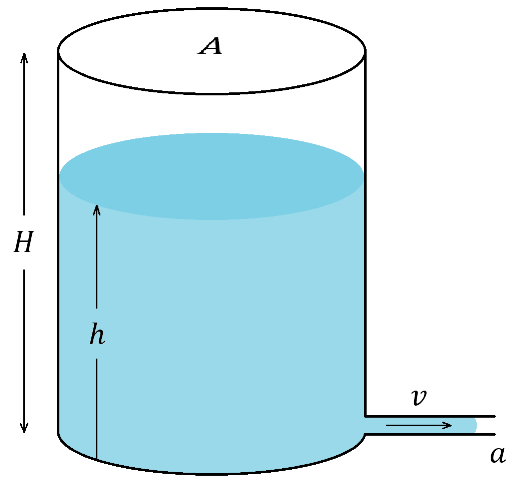
$$\theta_E \sim \sqrt{\frac{Gm(D - d)}{c^2 Dd}}.$$

You can repeat this argument, rotating in a circle around the line formed by the galaxy, star and observer on earth. We learn that the galaxy will appear as a ring, called an *Einstein ring*, of (angular) *Einstein radius* θ_E.

6. Explain why we don't observe Einstein rings around the sun. The sun has mass $m_\odot = 2 \times 10^{30}$ kg, radius $R_\odot = 7 \times 10^8$ m, and is $d = 150 \times 10^9$ m from earth.

2 It Doesn't Always Flow

A fully sealed water tank is initially filled up to height h_0 with the pressure of air inside being P_0, equal to the atmospheric pressure. An outlet located at the bottom of the tank is opened to slowly drain the water.



1. Assuming the draining process is slow enough for the system to maintain its temperature, find the velocity of water $v(h)$ exiting the outlet as a function of its height.

2. What volume of water has exited the tank after a long time?