Solutions to Problem Set No. 8

UBC Metro Vancouver Physics Circle 2018-2019

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A *Non*-Sticky Situation

1. Whenever there is collision involved, we can apply the law of conservation of momentum to solve for velocities of the system. In order to do this, however, we must find the pre-collision velocities since we are able to do so. Since we know that the pre-collision velocity of mass M is zero (at the peak of the projectile, right before collision), we only need to find the pre-collision velocity of mass m. By using one of the important equations of kinematics ($v^2 = v_0^2 + 2ad$), we can easily find the velocity of mass m as it displaces a height of h, which is found to be $v_{\text{peak}} = v_{1,i} = \sqrt{v^2 - 2gh}$. We can now set up our conservation of momentum equation within the $y-$direction.

$$\begin{aligned}
p_i &= p_f \\
mv_{1,i} + Mv_{2,i} &= mv_1 + Mv_2 \\
m\sqrt{v^2 - 2gh} + M(0) &= mv_1 + Mv_2 \\
\left(m\sqrt{v^2 - 2gh}\right)^2 &= \left(mv_1 + Mv_2\right)^2 \\
m^2(v^2 - 2gh) &= m^2v_1^2 + M^2v_2^2 + 2mMv_1v_2 \qquad \text{(Eqn. 1)}
\end{aligned}$$

The law of conservation of momentum is not able to solve for both v_1 and v_2 since there is only 1 equation and 2 variables. As a result, we must also use law of conservation of energy to introduce a second equation.

$$\begin{aligned}
E_{\text{total},i} &= E_{\text{total},f} \\
mgh + Mgh + \frac{1}{2}m(v_{1,i})^2 &= mgh + Mgh + \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2
\end{aligned}$$

$$m(v^2 - 2gh) = mv_1^2 + Mv_2^2$$

$$\frac{m(v^2 - 2gh) - Mv_2^2}{m} = v_1^2 \qquad \text{(Eqn. 2)}$$

By substituting Eqn. 2 into Eqn. 1:

$$\begin{aligned}
m^2(v^2 - 2gh) &= m^2(v^2 - 2gh) - mMv_2^2 + M^2v_2^2 + 2mMv_1v_2 \\
0 &= -mMv_2^2 + M^2v_2^2 + 2mMv_1v_2 \\
0 &= -mv_2^2 + Mv_2^2 + 2mv_1v_2 \\
0 &= (M - m)v_2^2 + 2mv_1v_2 \\
(m - M)v_2^2 &= 2mv_1v_2 \\
\Big((m - M)v_2\Big)^2 &= (2mv_1)^2 \\
(m - M)^2v_2^2 &= 4m^2v_1^2 \\
(m - M)^2v_2^2 &= 4m^2(v^2 - 2gh) - 4mMv_2^2 && \text{(Using Eqn. 2)} \\
(m^2 - 2mM + M^2)v_2^2 + 4mMv_2^2 &= 4m^2(v^2 - 2gh) \\
(m^2 + 2mM + M^2)v_2^2 &= 4m^2(v^2 - 2gh) \\
(M + m)^2v_2^2 &= 4m^2(v^2 - 2gh) \\
&\boxed{v_2 = \frac{2m\sqrt{v^2 - 2gh}}{M + m}} && \text{(We choose the } + \text{ root.)}
\end{aligned}$$

If we use Eqn. 2 and use our expression for v_2, we can now solve for v_1:

$$\begin{aligned}
v_1^2 &= \frac{m(v^2 - 2gh) - Mv_2^2}{m} \\
&= v^2 - 2gh - \frac{M}{m}\left(\frac{2m\sqrt{v^2 - 2gh}}{M + m}\right)^2 \\
&= v^2 - 2gh - \frac{M}{m}\left(\frac{4m^2(v^2 - 2gh)}{(M + m)^2}\right) \\
&= v^2 - 2gh - \frac{4mM(v^2 - 2gh)}{(M + m)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{4mM}{(M+m)^2}\right)(v^2 - 2gh) \\
&= \left(\frac{M^2 + 2mM + m^2 - 4mM}{(M+m)^2}\right)(v^2 - 2gh) \\
&= \frac{M^2 - 2mM + m^2}{(M+m)^2}(v^2 - 2gh) \\
&= \left(\frac{M-m}{M+m}\right)^2 (v^2 - 2gh) && \text{(Take the square root.)} \\
&= -\frac{M-m}{M+m}\sqrt{v^2 - 2gh} && \text{(We choose the $-$ root.)}
\end{aligned}
$$

$$
\boxed{v_1 = \frac{m-M}{M+m}\sqrt{v^2 - 2gh}}
$$

To find v_1 and v_2 in terms of m, M, v, and θ, we need to convert h into an expression involving θ. Since h is the height of the projectile's peak, we can use one of our kinematics equations to solve for h.

$$
\begin{aligned}
v^2 &= v_0^2 + 2ad \\
0 &= (v \sin\theta)^2 - 2gh \\
h &= \frac{v^2 \sin^2\theta}{2g} && \text{(Eqn. 3)}
\end{aligned}
$$

Using Eqn. 3 in the square root component of v_1 and v_2, we get:

$$
\begin{aligned}
\sqrt{v^2 - 2gh} &= \sqrt{v^2 - 2g\left(\frac{v^2 \sin^2\theta}{2g}\right)} \\
&= \sqrt{v^2 - v^2 \sin^2\theta} \\
&= v\sqrt{1 - \sin^2\theta} \\
&= v\sqrt{\cos^2\theta} \\
&= v\cos\theta
\end{aligned}
$$

Therefore, the v_1 and v_2 expressions become:

$$\boxed{v_1 = \frac{m - M}{M + m}\, v\cos\theta} \qquad \boxed{v_2 = \frac{2m}{M + m}\, v\cos\theta}$$

2. If $\alpha = \dfrac{m}{M}$, then the expression for v_2 becomes

$$v_2 = \frac{2m}{M+m}\, v\cos\theta = \frac{m}{M}\left(\frac{2}{1 + \frac{m}{M}}\right) v\cos\theta = \frac{2\alpha}{\alpha + 1}\, v\cos\theta$$

In order to find $f(\alpha, \theta)$, we must first solve for x, then D, then take a ratio.

$$\begin{aligned} &x \equiv (\text{horizontal velocity of mass } M) \times (\text{time it takes to hit the ground}) \\ &x = v_{2,x}\tau \end{aligned}$$

In the collision between m and M, the horizontal velocity of M is not impacted by m. Therefore, after the collision, $v_{2,x} = v\cos\theta$. We can use one of our primary kinematics equations to solve for τ, with the downwards direction being positive:

$$\begin{aligned} d &= v_0 t + \frac{1}{2}at^2 \\ h &= -v_2\tau + \frac{1}{2}g\tau^2 \\ 0 &= \frac{1}{2}g\tau^2 - v_2\tau - h \end{aligned}$$

We can now use the quadratic formula to solve for τ:

$$\begin{aligned} \tau &= \frac{v_2 \pm \sqrt{v_2^2 - 4\left(\frac{1}{2}g\right)(-h)}}{2\left(\frac{1}{2}g\right)} \\ &= \frac{v_2 \pm \sqrt{v_2^2 + 2gh}}{g} \\ &= \frac{v_2 + \sqrt{v_2^2 + 2gh}}{g} && \text{(We choose the } + \text{ root.)} \end{aligned}$$

$$= \frac{\dfrac{2\alpha}{\alpha+1}\ v\cos\theta + \sqrt{\left(\dfrac{2\alpha}{\alpha+1}\ v\cos\theta\right)^2 + 2g\left(\dfrac{v^2\sin^2\theta}{2g}\right)}}{g} \qquad \text{(Using } v_2\text{, Eqn. 3)}$$

$$= \frac{\dfrac{2\alpha\, v\cos\theta}{\alpha+1} + \sqrt{\dfrac{4\alpha^2 v^2\cos^2\theta}{(\alpha+1)^2} + v^2\sin^2\theta}}{g}$$

$$= \frac{\dfrac{2\alpha\, v\cos\theta}{\alpha+1} + v\cos\theta\sqrt{\dfrac{4\alpha^2}{(\alpha+1)^2} + \dfrac{\sin^2\theta}{\cos^2\theta}}}{g}$$

$$= \left(\frac{v\cos\theta}{g}\right)\left(\frac{2\alpha}{\alpha+1} + \sqrt{\frac{4\alpha^2}{(\alpha+1)^2} + \tan^2\theta}\right)$$

$$= \left(\frac{v\cos\theta}{g}\right)\left(\frac{2\alpha}{\alpha+1} + \frac{\sqrt{4\alpha^2 + (\alpha+1)^2\tan^2\theta}}{\sqrt{(\alpha+1)^2}}\right)$$

$$= \left(\frac{v\cos\theta}{g}\right)\left(\frac{2\alpha + \sqrt{4\alpha^2 + (\alpha+1)^2\tan^2\theta}}{\alpha+1}\right) \qquad \text{(Eqn. 4)}$$

As a result, we can use Eqn. 4 and the horizontal velocity to find x:

$$x = \left(\frac{v^2\cos^2\theta}{g}\right)\left(\frac{2\alpha + \sqrt{4\alpha^2 + (\alpha+1)^2\tan^2\theta}}{\alpha+1}\right)$$

Recall that the range, R, for any projectile that undergoes zero vertical displacement (shot at an angle θ above the horizontal) is equivalent to $\dfrac{v^2\sin(2\theta)}{g}$. Therefore, $D = \frac{1}{2}R$ which will be:

$$D = \frac{1}{2}R = \frac{v^2\sin(2\theta)}{2g} = \frac{2v^2\sin\theta\cos\theta}{2g} = \frac{v^2\sin\theta\cos\theta}{g}$$

Combining the two, we can solve for $f(\alpha, \theta)$:

$$f(\alpha, \theta) = \frac{x}{D}$$

$$f(\alpha, \theta) = \frac{\left(\dfrac{v^2 \cos^2 \theta}{g}\right)}{\left(\dfrac{v^2 \sin \theta \cos \theta}{g}\right)} \left(\frac{2\alpha + \sqrt{4\alpha^2 + (\alpha + 1)^2 \tan^2 \theta}}{\alpha + 1}\right)$$

$$\boxed{f(\alpha, \theta) = \frac{2\alpha + \sqrt{4\alpha^2 + (\alpha + 1)^2 \tan^2 \theta}}{(\alpha + 1) \tan \theta}}$$

3. Since $\tan\left(\frac{\pi}{4}\right) = 1$ and our ratio function is equivalent to 2, this simplifies it down to:

$$2 = \frac{2\alpha + \sqrt{4\alpha^2 + (\alpha + 1)^2}}{\alpha + 1}$$

$$2\alpha + 2 - 2\alpha = \sqrt{4\alpha^2 + \alpha^2 + 2\alpha + 1}$$

$$2 = \sqrt{5\alpha^2 + 2\alpha + 1}$$

$$4 = 5\alpha^2 + 2\alpha + 1$$

$$0 = 5\alpha^2 + 2\alpha - 3$$

$$\alpha = \frac{-2 \pm \sqrt{4 - 4(5)(-3)}}{10}$$

$$\alpha = \frac{-2 \pm 8}{10} = -1, \frac{3}{5}$$

Therefore, the realistic solution is $\boxed{\alpha = \frac{3}{5}}$.

4. This time, we are not given θ but the ratio function is still equal to 2. Let's try to solve for α given the angle variable θ. The steps will be similar to part (3).

$$
\begin{aligned}
2 &= \frac{2\alpha + \sqrt{4\alpha^2 + (\alpha+1)^2 \tan^2\theta}}{(\alpha+1)\tan\theta} \\
2(\alpha+1)\tan\theta - 2\alpha &= \sqrt{4\alpha^2 + (\alpha+1)^2 \tan^2\theta} \\
\Big(2(\alpha+1)\tan\theta - 2\alpha\Big)^2 &= 4\alpha^2 + (\alpha+1)^2 \tan^2\theta \\
4(\alpha+1)^2\tan^2\theta - 8\alpha(\alpha+1)\tan\theta + 4\alpha^2 &= 4\alpha^2 + (\alpha+1)^2\tan^2\theta \\
3(\alpha+1)^2\tan^2\theta - 8\alpha(\alpha+1)\tan\theta &= 0 \\
3(\alpha+1)\tan\theta - 8\alpha &= 0 \\
\alpha(3\tan\theta - 8) + 3\tan\theta &= 0 \\
\alpha &= \frac{3\tan\theta}{8 - 3\tan\theta}
\end{aligned}
$$

Since earlier we divided by a $(\alpha+1)\tan\theta$, we have restrictions of $\alpha + 1 \neq 0$ and $\tan\theta \neq 0$. The $\alpha \neq -1$ restriction is not possible, so we do not have to worry about it. Also, the $\tan\theta \neq 0$ restriction leads to $\theta \neq 90°, 270°, ...$ which again is not a concern. However, the α expression has a denominator. As a result:

$$
\begin{aligned}
8 - 3\tan\theta &\neq 0 \\
\tan\theta &\neq \frac{8}{3} \\
\theta &\neq \frac{180°}{\pi}\arctan\left(\frac{8}{3}\right) = 69.44°
\end{aligned}
$$

*Note: We have inserted the term $\frac{180°}{\pi}$ because we want to calculate our angle in degrees.

Therefore, $\boxed{\phi = 69.44°}$ where $0° < \theta < 69.44°$ are the only valid angles that allow the ratio function to equal to 2 given some α value. Mathematically, it is true that $\alpha(\theta)$ exists for angles $69.44° < \theta < 90°$, but because of the discontinuity at $\theta = 69.44°$ and the nature of the tangent function, α values in this range will be negative. In other words, these α values cannot exist. Ultimately, this means that no matter what mass ratio you choose in an experiment (choosing your α value), you can never achieve a ratio function of 2 if the θ is initially set at an angle above $\phi = 69.44°$.

5. We will repeat part (4), but have the ratio function equal to a value c, where $c > 1$.

$$\begin{aligned}
c &= \frac{2\alpha + \sqrt{4\alpha^2 + (\alpha+1)^2 \tan^2\theta}}{(\alpha+1)\tan\theta} \\
c(\alpha+1)\tan\theta - 2\alpha &= \sqrt{4\alpha^2 + (\alpha+1)^2 \tan^2\theta} \\
\Big(c(\alpha+1)\tan\theta - 2\alpha\Big)^2 &= 4\alpha^2 + (\alpha+1)^2 \tan^2\theta \\
c^2(\alpha+1)^2\tan^2\theta - 4c\alpha(\alpha+1)\tan\theta + 4\alpha^2 &= 4\alpha^2 + (\alpha+1)^2\tan^2\theta \\
(c^2-1)(\alpha+1)^2\tan^2\theta - 4c\alpha(\alpha+1)\tan\theta &= 0 \\
(c^2-1)(\alpha+1)\tan\theta - 4c\alpha &= 0 \\
\alpha\Big((c^2-1)\tan\theta - 4c\Big) + (c^2-1)\tan\theta &= 0 \\
\alpha &= \frac{(c^2-1)\tan\theta}{4c - (c^2-1)\tan\theta}
\end{aligned}$$

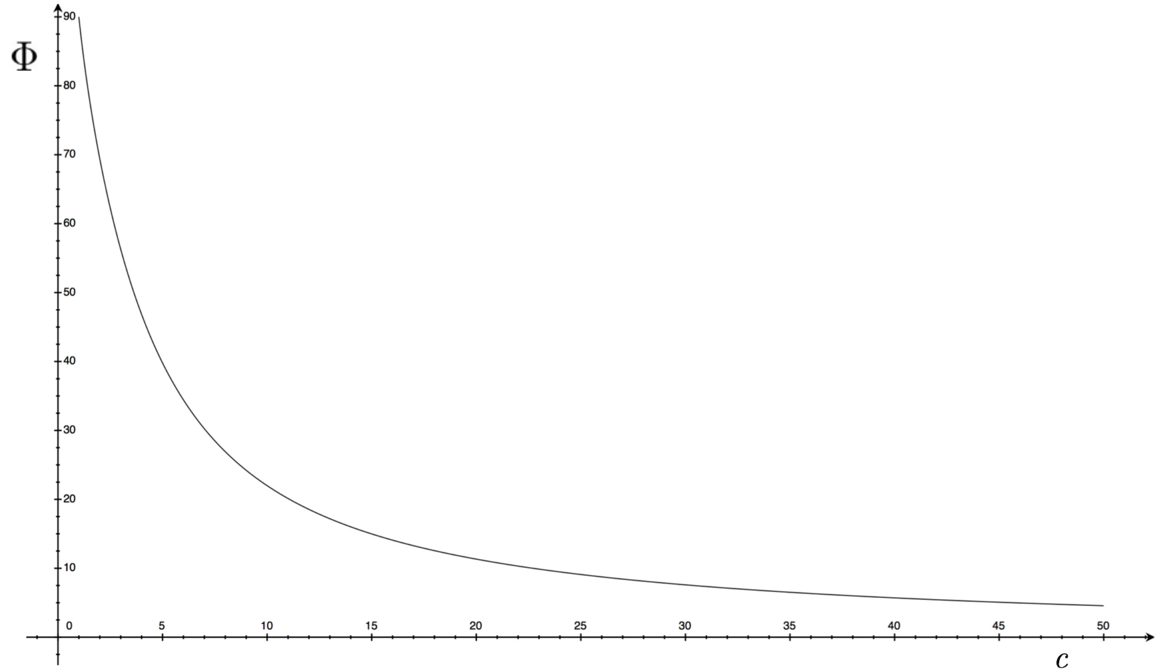
As in part (4), we divided by a $(\alpha+1)\tan\theta$, and thus we have restrictions of $\alpha+1 \neq 0$ and $\tan\theta \neq 0$. Reiterating from part (4), both restrictions are not of concern. However, the α expression has a denominator. As a result:

$$\begin{aligned}
4c - (c^2-1)\tan\theta &\neq 0 \\
\tan\theta &\neq \frac{4c}{c^2-1} \\
\theta &\neq \arctan\left(\frac{4c}{c^2-1}\right)
\end{aligned}$$

Therefore, inserting a $\dfrac{180°}{\pi}$ term to calculate our angle in degrees,

$$\boxed{\Phi(c) = \frac{180°}{\pi}\arctan\left(\frac{4c}{c^2-1}\right)}$$

Below is the graph of $\Phi(c)$ for $1 < c < 50$:



To reiterate from part (4), it is true that mathematically $\alpha(\theta)$ exists for angles in the range $\Phi(c) < \theta < 90°$. However, because of the discontinuity at $\theta = \Phi(c)$ and the nature of the tangent function, α values in this range will be negative. In other words, these α values cannot exist. Ultimately, this means that no matter what mass ratio you choose, you can never achieve a ratio function of value c if the θ is initially set at an angle above $\Phi(c)$.

6. Recall that

$$v_2 = \frac{2\alpha}{\alpha + 1}\, v \cos\theta$$

For simplicity, let's call $k = \dfrac{2\alpha}{\alpha+1}$ such that $v_2 = kv\cos\theta$. We know from Eqn. 3 that $h = \dfrac{v^2 \sin^2\theta}{2g}$. However, M travels an extra height δ during the second projectile, such that its height from the ground at its vertex is some variable $Y = h + \delta$. Let's find δ:

$$v^2 = v_0^2 + 2ad$$

$$0 = k^2 v^2 \cos^2\theta - 2g\delta$$

$$2g\delta = k^2 v^2 \cos^2\theta$$

$$\delta = \frac{k^2 v^2 \cos^2\theta}{2g}$$

If the maximum vertical height of the second m is H, we can similarly use the above kinematics equation to find that $H = v^2/2g$. Finding the difference between H and Y will determine if there will be a second collision:

$H - Y = \oplus \quad \longrightarrow$ This indicates a guaranteed hit.

$H - Y = 0 \quad \longrightarrow$ This will lead to contact, but no transfer of momentum (i.e. no collision).

$H - Y = \ominus \quad \longrightarrow$ This indicates no hit.

Therefore, the condition that we want is to have $H - Y > 0$. Let's solve for this:

$$\begin{aligned}
H - Y = H - (h + \delta) &> 0 \\
\frac{v^2}{2g} - \left(\frac{v^2\sin^2\theta}{2g} + \frac{k^2v^2\cos^2\theta}{2g}\right) &> 0 \\
\frac{v^2}{2g} - \frac{v^2}{2g}\left(\sin^2\theta + k^2\cos^2\theta\right) &> 0 \\
\frac{v^2}{2g}\Big(1 - \left(\sin^2\theta + k^2\cos^2\theta\right)\Big) &> 0 \\
\frac{v^2}{2g}\left(1 - \sin^2\theta - k^2\cos^2\theta\right) &> 0 \\
\frac{v^2}{2g}\left(\cos^2\theta - k^2\cos^2\theta\right) &> 0 \\
\frac{v^2\cos^2\theta}{2g}\left(1 - k^2\right) &> 0
\end{aligned}$$

Since the term $\dfrac{v^2\cos^2\theta}{2g}$ will always be positive, the $(1 - k^2)$ term is the only one that will be affected by the inequality.

$$\begin{aligned}
1 - k^2 &> 0 \\
k^2 &< 1
\end{aligned}$$

This will lead to $-1 < k < 1$, but values in the $-1 < k < 0$ are not valid as that will lead to a negative α value. As a result, the range of values will be $0 < k < 1$. This will lead to α values in the range:

$$\begin{aligned} 0 < & \quad k & < 1 \\ 0 < & \frac{2\alpha}{\alpha+1} & < 1 \\ 0 < & \quad 2\alpha & < \alpha+1 \\ 0 < & \quad \alpha & < 1 \end{aligned}$$

As a result, $\boxed{0 < \alpha < 1}$ is the only range of α values that guarantees a second collision between M and m.