UBC Physics Circle

Session 1: Problems

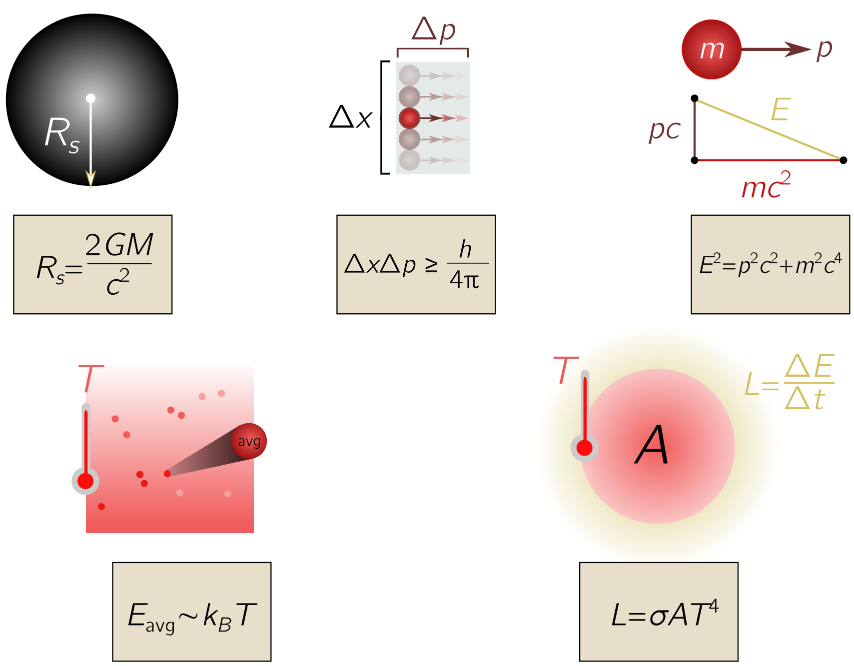
October 3, 2019

For our first problem session of 2019–20, we will follow in the illustrious footsteps of Stephen Hawking, and discover that black holes *glow*. This means they slowly leak energy into space, and eventually vanish in a burst of high-energy radiation! We will use this to calculate the lifetime of a solar mass black hole.

We're going to need five basic facts from Douglas Scott's talk. These are:

- the size of a black hole;
- the uncertainty principle of quantum mechanics;
- the relativistic energy of a moving particle;
- the average energy of particles in hot systems; and
- the rate a hot object radiates energy.

We picture these using cartoons below.



Let's go through these more carefully:

- **Black holes.** First, a black hole of mass M has a *Schwarzschild radius* of

$$R_s = \frac{2GM}{c^2}, \tag{1}$$

where G is *Newton's gravitational constant*, and c is the speed of light, given by

$$G = 6.67 \times 10^{-11}\,\text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2} \tag{2}$$

$$c = 3.00 \times 10^{8}\,\text{m s}^{-1}. \tag{3}$$

If you venture within R_s of the black hole, you can never escape! You will get a chance to derive this below.

- **Uncertainty.** A second useful fact is *Heisenberg's uncertainty principle*, which tells us that the uncertainty in position, multiplied by the uncertainty in momentum, are at least as big as *Planck's constant* h:

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}, \quad h = 6.63 \times 10^{-34}\,\text{J s}^{-1}. \tag{4}$$

For classical measurements, this is typically much smaller than the precision of our instruments, but this tiny, inescapable uncertainty has very deep consequences.

- **Relativistic energy.** Third, we need a generalisation of Einstein's famous relation $E = mc^2$, which encompasses both light and matter:

$$E^2 = m^2c^4 + p^2c^2. \tag{5}$$

Here, m is the *rest mass*, and p is the usual classical momentum. This reduces to $E = mc^2$ for a stationary object, but gives

$$E = pc \tag{6}$$

for an object like a photon which has no rest mass.

- **Thermal energy.** Fourth on our list is the connection between temperature and typical energy. For a lump of particles (including massless particles like photons!) at temperature T, the *typical energy per particle* is

$$E_{\text{avg}} \sim k_B T, \quad k_B = 1.38 \times 10^{-23}\,\text{J K}^{-1}, \tag{7}$$

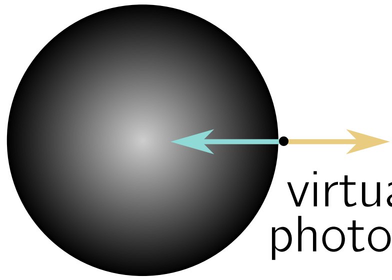
where k_B is *Boltzmann's constant*, not to be confused with the (related) Stefan-Boltzmann constant governing luminosity. There are various dimensionless constants that can appear in (7), related to the properties of the system, but we won't need them here.

- **Luminosity.** Finally, we require the *Stefan-Boltzmann law*. This tells us that a hot object at temperature T (in Kelvin), with surface area A, loses energy at a rate L (for *luminosity*) given by

$$L = \sigma A T^4, \quad \sigma = 5.67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}, \tag{8}$$

where σ is the *Stefan-Boltzmann constant*.[1]

In 1974, Stephen Hawking discovered a remarkable fact: black holes glow, emitting faint radiation like a lump of coal or a light bulb at some temperature T. We can determine the temperature T, and hence the rough time scale for evaporation, but first we should explain how it is even *possible* for a black hole to glow, when by definition it traps light.



The heuristic explanation is as follows: quantum mechanics allows for the production of *virtual pairs of photons* moving in opposite directions. Usually, these pop into existence briefly and then disappear again. But just outside the black hole, one of these photons can fall into the event horizon, while the other zooms off to infinity! It is this second photon that we can detect.

1. Suppose a small particle of mass m is a distance r from the black hole, mass M. The *gravitational potential energy* is
$$U = -\frac{GMm}{r}.$$
This is negative, since we must put energy U into the system to separate the mass m and the black hole so that they no longer feel each other's gravitational influence.

   (a) Suppose we try to separate the particle from the black hole by giving it some kinetic energy. Show that, starting at distance r, it must travel at speed
   $$v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$$
   in order to escape the black hole's gravitational pull.

   (b) As we get closer to the black hole, the escape velocity will increase. Show that at the Schwarzschild radius (1), the escape velocity becomes the speed of light. Not even photons can escape!

2. (a) For particles produced near the horizon of a black hole, what is the rough uncertainty in position? Don't worry about numerical factors.

   (b) Using the Heisenberg uncertainty principle (4), argue that the uncertainty in momentum is order
   $$\Delta p \sim \frac{hc^2}{GM}.$$

[1] Depending on properties of the body and its surrounding medium, the rate L can be modified by an *emissivity factor* ϵ, $0 < \epsilon \leq 1$. A *perfect blackbody* (perfect absorber and emitter) has $\epsilon = 1$. A black hole is, classically, a perfect absorber, and once we take quantum mechanics into account, a perfect emitter too!

3. (a) From Einstein's relation (6), find the uncertainty in energy ΔE. Assuming that the uncertainty in energy is roughly the same as the energy of a typical virtual photon,[2]

$$E_{\text{avg}} \sim \Delta E,$$

conclude that

$$E_{\text{avg}} \sim \frac{hc^3}{GM}.$$

(b) From (7), deduce that the Hawking temperature of a black hole of mass M is

$$T_H \sim \frac{hc^3}{k_B GM}. \tag{9}$$

Notice that the black hole gets hotter as it gets smaller!

4. (a) What is the total energy content of a black hole, of mass M, at rest? *Hint.* Use (5).

(b) The luminosity is the rate of energy loss. Argue (from dimensional analysis or otherwise) that the lifetime of a black hole is

$$t_{\text{life}} \sim \frac{Mc^2}{L} = \frac{Mc^2}{\sigma A T_H^4}.$$

(c) Use $A \sim R_s^2$ and (9) to show that

$$t_{\text{life}} \sim \frac{G^2 k_B^4 M^3}{\sigma h^4 c^6}. \tag{10}$$

(d) The sun has mass

$$M_\odot = 2 \times 10^{30} \text{ kg}.$$

Plug this into (10), along with values for the various fundamental constants, and estimate the lifetime of a solar-mass black hole. Give your answer in years, and compare to the age of the universe, $\sim 10^{10}$ y.

[2]This is a weird property. Usually, the average energy E_{avg} is completely unrelated to the uncertainty ΔE. It just so happens that, for a hot gas of photons, this relation $E_{\text{avg}} \sim \Delta E$ is true. So our assumption is really that the virtual photons form a gas.

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1. (a) The kinetic energy for the particle is $E = mv^2/2$. The escape velocity v_{esc} is the smallest velocity needed to escape the black hole, with $E + U = 0$, or

$$0 = \frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{r} \quad \implies \quad v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.$$

   (b) We simply set $v_{\text{esc}} = c$, $r = R_S$, and make R_S the subject:

$$c = \sqrt{\frac{2GM}{R_s}} \quad \implies \quad R_s = \frac{2GM}{c^2}.$$

   This is the Schwarzschild radius, as promised!

2. (a) They can be produced anywhere on the surface of the black hole. This has area $A = 4\pi R_s^2$, but the uncertainty Δx should have the dimensions of length, not area (length squared). So we just take the square root:

$$\Delta x \sim \sqrt{A} \sim R_s.$$

   In other words, the position uncertainty is the Schwarzschild radius.[3]

   (b) Let's assume that the uncertainty is close to the minimum allowed by quantum mechanics. This turns out to be reasonable, since virtual particles are only possible by virtue of the uncertainty principle! Ignoring numerical factors, this means

$$\Delta p \sim \frac{h}{\Delta x} \sim \frac{h}{R_s} \sim \frac{hc^2}{GM}.$$

3. (a) If the energy of a photon is related to its momentum by $E = pc$, then the uncertainty is just

$$\Delta E = \Delta(pc) = c\Delta p,$$

   since the speed of light is a constant. From the assumption that $E_{\text{avg}} \sim \Delta E$, we have

$$E_{\text{avg}} \sim c\Delta p \sim \frac{hc^3}{GM}.$$

   (b) Using (7), the Hawking temperature is

$$T_H \sim \frac{E_{\text{avg}}}{k_B} \sim \frac{hc^3}{k_B GM}.$$

[3] If we assume that the uncertainty is precisely half the circumference, we exactly reproduce Hawking's answer! But we cut so many corners in getting the "correct" answer that it is a bit of a coincidence.

4. (a) A black hole at rest obeys Einstein's most famous equation,

$$E = Mc^2.$$

(b) The lifetime should the *total energy* divided by the *rate of at which energy is lost.*[4] The previous question gives the total energy, and the luminosity is given by (8). Hence, we have an estimate for the lifetime

$$t_{\text{life}} \sim \frac{E}{L} \sim \frac{Mc^2}{\sigma A T_H^4}.$$

The luminosity will change as the black hole loses energy, but for a rough sense of the time scale, this estimate is reasonable.

(c) Let's simplify the lifetime from the previous question. We will use $A \sim R_s^2$, (1), and (9). After some algebram

$$t_{\text{life}} \sim \frac{Mc^2}{\sigma R_s^2 T_H^4} \sim \frac{Mc^6}{\sigma (GM)^2} \cdot \left(\frac{hc^3}{k_B GM}\right)^{-4} = \frac{G^2 k_B^4 M^3}{\sigma h^4 c^6}.$$

This is what we were aiming for!

(d) Let's plug in numbers and see what we get:

$$\begin{aligned} t_{\text{life}} &\sim \frac{G^2 k_B^4 M^3}{\sigma h^4 c^6} \\ &= \frac{(6.67 \times 10^{-11})^2 (1.38 \times 10^{-23})^4 (2 \times 10^{30})^3}{(5.67 \times 10^{-23})(6.63 \times 10^{-34})^4 (3 \times 10^8)^6} \text{ s} \\ &\approx 1.62 \times 10^{68} \text{ s} \\ &\approx 5 \times 10^{60} \text{ y.} \end{aligned}$$

This is many, many orders of magnitude larger than the age of the universe! A solar mass black hole is likely to be around long after all the stars in the universe have burnt themselves out, like candles in an empty house, glowing more brightly as they get smaller.

[4] If this seems confusing, consider the analogous statement about distance, time and velocity: the time taken to get from A to B is the total distance divided by the velocity (rate at which distance is gained).