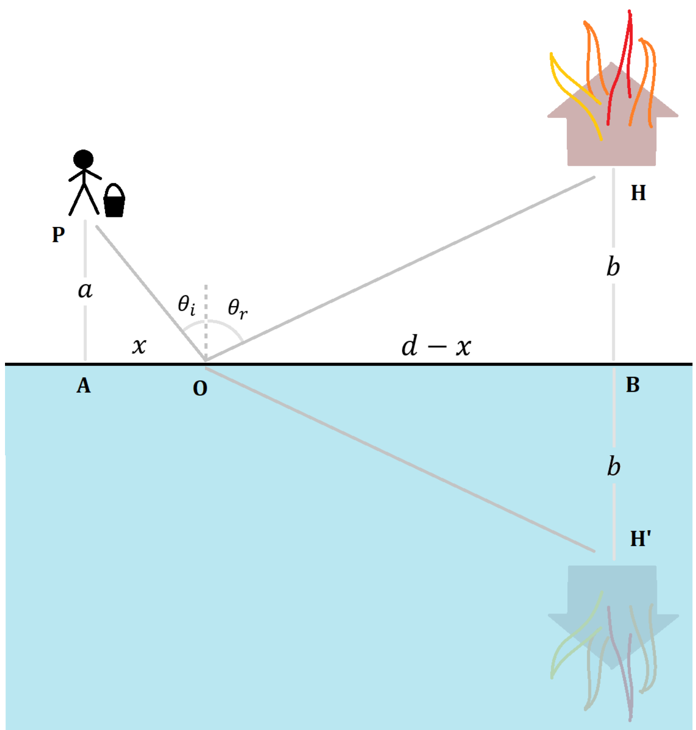
UBC Physics Circle

Session 4: Lecture Solutions

November 16, 2019

House on Fire

The trick is to imagine walking towards the image of the house about the shoreline instead. In the figure below, we have for the lengths $\overline{OH} = \overline{OH'}$ and therefore $\overline{POH} = \overline{POH'}$. Hence, minimizing $\overline{POH'}$ identically minimizes $\overline{POH}$.



Since the path of least time is the shortest path in this case (there is no change of medium, and hence speed), the $\overline{POH'}$ resulting in least time is a straight line.
An important consequence of this is that the angles $P\widehat{O}A = H'\widehat{O}B = H\widehat{O}B$ and therefore $\theta_i = \theta_r$, which is the law of reflection. This problem is an analogy for the path light would take when reflecting, which is the path of least time according to Fermat's principle.

From there, to find x, we can use the similarity of triangles $P\overset{\triangle}{O}A \sim H\overset{\triangle}{O}B$ to write:

$$\frac{b}{a} = \frac{d-x}{x} \implies x = \frac{d}{1+b/a}$$

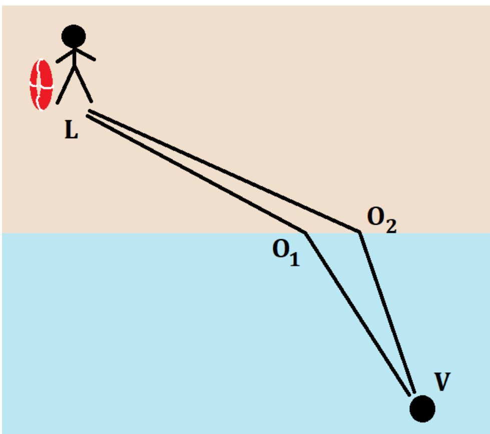
Speedy Rescue

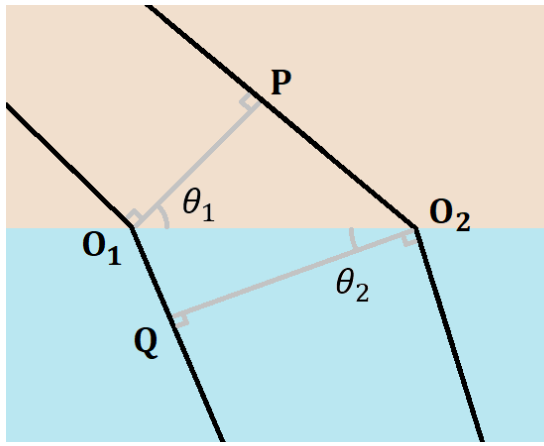
For some relation between the angles θ_1 and θ_2, we obtain the path of least time (called best path hereafter). To find this relation, let us look at how the total travel time varies as we make slight changes to the path.

As depicted in the figure on left, consider the two paths $\overline{LO_1V}$ and $\overline{LO_2V}$ for some small change $\overline{O_1O_2}$. Close to the best path, small changes have negligible effect on the total travel time (this is a deep statement, think about its validity). Therefore, to find the best path, we require:

$$t_{LO_1P} = t_{LO_2P}$$

The labels are from figure on right, which provides a magnified view of the comparison.





Assuming $\overline{O_1O_2}$ to be small and noting that $t_{LO_1} = t_{LP}$ and $t_{QV} = t_{O_2V}$, we have:

$$t_{O_1Q} = t_{PO_2}$$

And with some trigonometry, we arrive at the answer:

$$\frac{\overline{O_1O_2}\sin(\theta_1)}{v_1} = \frac{\overline{O_1O_2}\sin(\theta_2)}{v_2} \implies \frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}$$

This problem is an analogue of light refraction. The path of least time when changing media is not a straight line. Our finding above takes a more familiar look by setting $v_1 = c/n_1$ and $v_2 = c/n_2$, the speed of light in each medium. The result is Snell's law of refraction:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Wow! Did we just derive Snell's law using simple trigonometry? Well, we used some very non-trivial concepts along the way. Make sure you go through the solution again to better understand the thought process.

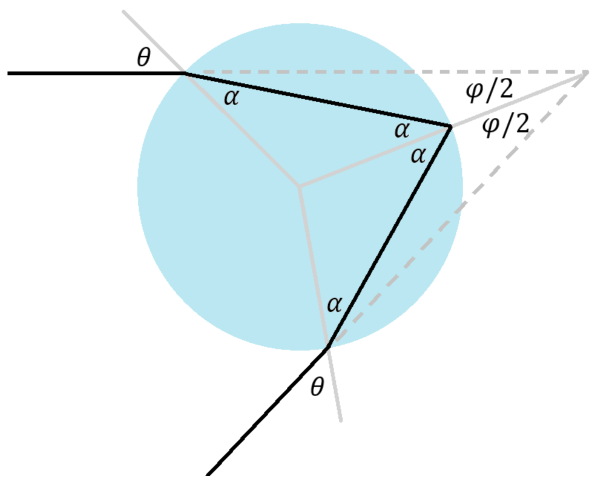
The Primary Bow

Taking θ to be the angle of incidence, we know from Snell's law that the refracted angle α is given by:

$$\alpha = \sin^{-1}\left(\frac{\sin(\theta)}{n}\right)$$

where n is the refractive index of water for a given wavelength of light.
Using some geometry, we label angles as shown below and write the sum of angles for one of the large triangles with angle $\varphi/2$:

$$\frac{\varphi}{2} + \theta + (\pi - 2\alpha) = \pi \implies \varphi = 4\alpha - 2\theta = 4\sin^{-1}\left(\frac{\sin(\theta)}{n}\right) - 2\theta$$



The next step to seeing how this relation results in the primary bow is to plot it. You can see plots of above relation $\varphi(\theta)$ for the different indices of refraction of red, green, and violet light at desmos.com/calculator/koohasiyly.

Also, here is the link to the prism interactive we saw desmos.com/calculator/chntfgzgol.