Dimensional analysis

Physics circle 2021

What do we mean by dimension?

- A dimension is a physical quantity that can be measured, like length or mass.
- A measurement of a physical quantity is expressed in terms of units, which are standardized values. All units of the same dimension are related through a conversion factor. For instance, I can measure the length of my pencil with a ruler in centimeters. Here, the dimension is length and the units are centimeters, and a possible conversion factor is 100cm=1m.

Base quantities

Base Quantity	Symbol for Dimension
Length	L
Mass	M
Time	T
Current	I
Thermodynamic temperature	Θ
Amount of substance	N
Luminous intensity	J

The dimension of any measurement is dependent on these base quantities. Dimensions obey the rules of algebra. So, for example, area is the product of two lengths, so it has dimension L^2. and similarly, volume has dimension L^3.

We usually use square brackets to refer to the dimension of a quantity. For example, we would write [V]=L^3, where [V] means the dimension of volume. This is just notation.

Why are dimensions important?

The importance of dimension is that any equation relating physical quantities has to be dimensionally consistent, meaning:

a) Every term in the equation must have the same dimension, you can't add apples and oranges just like you can't add seconds and meters.
b) The arguments of Standard mathematical functions like sines and cosines must be dimensionless: inputs are scalars and outputs are scalars.

Dimensional analysis helps us to verify if our equations are correct!

Some practice!

Consider: the quantities s, v, a, and t with dimensions [s]= L, [v]=LT^-1 , [a]=LT^-2, and [t]=T.

Are the following equations dimensionally consistent?

a) s=vt+0.5at^2
b) s=vt^2+0.5at

a) $s=vt+0.5at^2$

$[vt]=[v][t]=(LT^{-1})(T)=L$

$[0.5at^2]=[a][t]^2=(LT^{-2})(T^2)=L$

So this is dimensionally consistent!

Some practice!

Consider: the quantities s, v, a, and t with dimensions $[s] = L$, $[v] = LT^{-1}$, $[a] = LT^{-2}$, and $[t] = T$.

Are the following equations dimensionally consistent?

a) $s = vt + 0.5at^2$
b) $s = vt^2 + 0.5at$

b) $s = vt^2 + 0.5at$

$[s] = L$

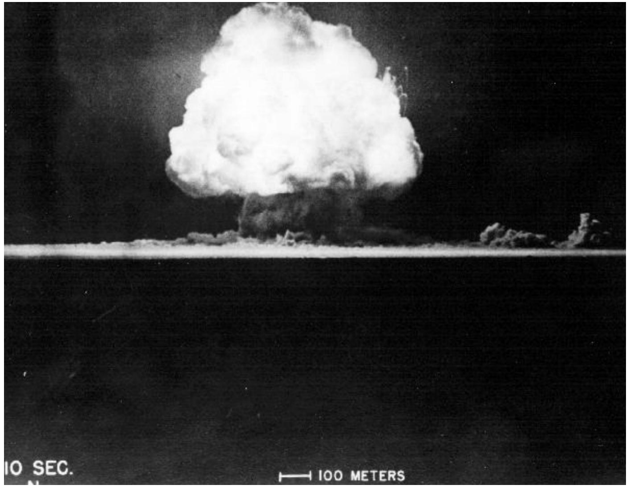
$[vt^2] = [v][t]^2 = (LT^{-1})T^2 = LT$

$[at] = [a][t] = (LT^{-2})(T) = LT^{-1}$

This one is not dimensionally consistent :(

Problem: The first atomic bomb.

The first atomic bomb was detonated on july 16, 1945, at the trinity test site, about 200 mi south of Los Alamos. In 1947, the U.S government decided to release a film reel of the explosion. From this film, the british physicist G.I. Taylor was able to determine the rate at which the radius of the fireball from the blast grew. Using dimensional analysis, he was then able to deduce the amount of energy released in the explosion, which was a closely guarded secret at the time. Because of this, Taylor did not publish his results until 1950. This problem challenges you to recreate this famous calculation.



First of all, let’s try to make an educated guess.

Based on your intuition, The radius of the fireball formed after the explosion should depend on:

a) energy released on the explosion and distance from the from the center of the explosion.

b) density of the air and time since the explosion.

c) temperature of the air and distance from the center of the explosion.

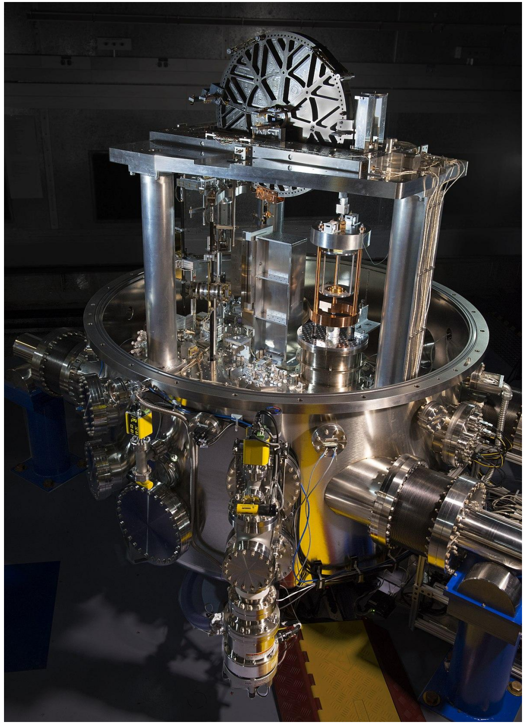
d) energy released, density of the air and time since the explosion.

Taylor noticed that the radius r of the fireball should depend only on time since the explosion, t, the density of the air, ρ, and the energy of the initial
explosion, E. Thus, he made the educated guess that
r = kE^a ρ^bt^c for some dimensionless constant k and some
unknown exponents a, b, and c. Given that [E] = ML^2T^-2, determine the values of the exponents necessary to make this equation dimensionally consistent. Note that k has dimension 1.

$$r = kE^{a}\rho^{b}t^{c}$$

The kibble balance

In 2018, the definition of the kilogram was redefined using the kibble balance. The kibble balance, (originally named Watt balance) was designed by Bryan Kibble in 1995 and it is a device that allows us to relate mass to a very important physical constant: Planck’s constant.



If we write a relationship for a mass that is measured with a kibble balance, we will find that the equation contains the parameters g, ε, and I. We can measure g with high precision using an absolute gravimeter while the induced emf(ε)and and current (I) can be measured using Josephson's effect and the quantum Hall effect, respectively.

First of all, Josephson's effect measures a Josephson's voltage between two superconductor materials that are separated by a non superconductor material. Then the following relationship is satisfied:

$$V_J = \frac{f}{K_J}$$

Where Vj is Josephson's voltage and Kj is Josephson's constant.
Using the fact that Planck's constant h=6.63x10^34 Js and the charge of the electron is e=1.6x10^-19 C, find an expression of Kj in terms of the charge of the electron and Planck's constant if there is a proportionality factor of 2.

Optional: Now let's talk about the quantum Hall effect

On the other hand, the electrical Hall resistance (Rh) of the superconductor material satisfies the relationship:

$$R_H = \frac{1}{n} R_K$$

Here, Rk is Von Klitzing constant and n is an integer. Now find find an expression for Rk in terms of Planck's constant (h) and the charge of the electron (e).

Solution Atomic bomb

1. Write the dimension of each of the quantities:

$$[R] = L$$

$$[E] = \frac{ML^2}{T^2}$$

$$[t] = T$$

$$[\rho] = \frac{M}{L^3}$$

2. We then can set the equation:

$$L = \left(\frac{ML^2}{T^2}\right)^a \times T^b \times \left(\frac{M}{L^3}\right)^c$$

3. This gives us a system of three equations!

$$1 = 2a - 3c$$

$$0 = -2a + b$$

$$0 = a + c$$

4. We solve the system to find:

$$a = \frac{1}{5} \quad ; \quad b = \frac{2}{5} \quad , \quad c = -\frac{1}{5}.$$

Solution Kibble balance

We start by writing the dimensions of the quantities:

$$[V_J] = \frac{ML^2}{qT^2}$$

$$[f] = \frac{1}{T}$$

$$[e] = q$$

$$[h] = \frac{ML^2}{T}$$

$$[K_J] = ?$$

From:

$$V_J = \frac{f}{K_J}$$

We then set the equation:

$$\frac{ML^2}{qT^2} = \frac{1}{[K_J]T}$$

Finally, we solve for the dimension of kj to find:

$$[K_J] = \frac{qT}{ML^2} = \frac{2e}{h}.$$

(We shouldn't forget our proportionality factor of 2!)

How cool is that! We just found Josepshon's constant in terms of two fundamental constants :)