A Tale of Two Black Holes

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UBC Physics Circle

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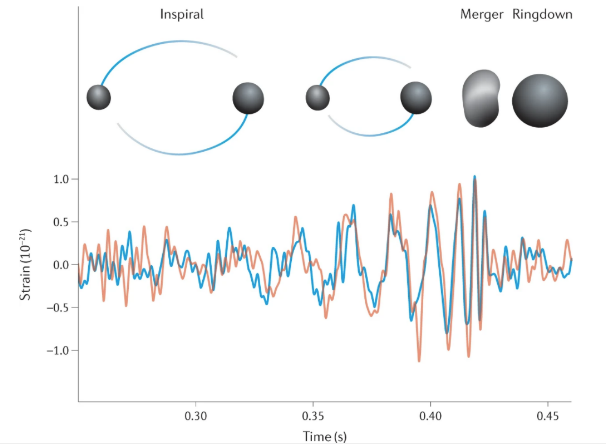

Figure 1: The waveforms are shifted and inverted to compensate for the slightly different arrival times and different orientations of the detectors (red: LIGO Hanford, blue: LIGO Livingston). LIGO data and simulated evolution of the black hole event horizons as the system coalesces and merges.

Our goals:

1. Estimate how much energy is released during the collision.
2. Estimate the energy that arrived at Earth from this collision.
3. Determine the distortion on Earth and its frequency.
4. Identify why gravitational waves are so hard to measure.

What is the **initial energy** from two black holes of mass M_1, M_2?
(Assume they are infinitely far away.)

A. $\frac{M_1 M_2}{M_1+M_2} c^2$
B. $(M_1 + M_2)c^2$
C. $\sqrt{M_1^2 + M_2^2} c^2$
D. $\sqrt{M_1 M_2} c^2$

Energy Released from Colliding Black Holes

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D. $\sqrt{M_1 M_2}c^2$

Black Hole Final Energy

- Second law of Thermal Dynamics: entropy (chaos) of a system is always increasing.
- Hawking: Entropy of a black hole is proportional to the surface area, hence the **surface area must be increasing**.

Black Hole Area Theorem

$$A_{final} \geq A_{1,init} + A_{2,init}$$

To get the maximum energy transmitted, we want to "lose" the most area, so we will take the quality:

$$A_{final} = A_{1,init} + A_{2,init}$$

Schwartzschild Radius

Recall in the first session, we derived the Schwartzschild radius, which is the radius of an event horizon for a Schwartzschild black hole.

$$R_B = 2\frac{GM}{c^2}$$

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Exercise:

1. Show the expression $R = \frac{GM}{c^2}$ is the radius for any mass m orbiting around a central mass M.
2. Show this expression is dimensionally consistent.

Using the Schwartzschild radius of a Black Hole:

$$R_B = 2\frac{GM}{c^2}$$

Rewrite the area theorem into a statement of masses.

$$A_{final} = A_{1,init} + A_{2,init}$$
$$4\pi R_{B,final}^2 = 4\pi R_{B,1,init}^2 + 4\pi R_{B,2,init}^2$$

Simplifying the expression, we will find that:

$$M_{final}^2 = M_1^2 + M_2^2$$

What is the **minimum final energy** after the collision between two black holes of mass M_1, M_2?

A. $\frac{M_1 M_2}{M_1+M_2} c^2$

B. $(M_1 + M_2)c^2$

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Black Hole Final Mass

$$M_f^2 = M_1^2 + M_2^2$$

Suppose we have two black holes of equal masses: $M_1 = M_2 = M$. Based on the relation of final masses, what is the energy of the black hole formed by their collision?

A $\sqrt{2}Mc^2$

B Mc^2

C $2Mc^2$

D $4Mc^2$

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Black Hole Energies

Recall our calculations of the energies for two colliding black holes of masses $M_1 = M_2 = M$:

$$E_i = 2Mc^2$$
$$E_f = \sqrt{2}Mc^2$$

What is the maximum percentage of energy lost during this collision?

A. 29%

B. 41%

C. 71%

D. 100%

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Exercise: So far we concluded 29% of the original black holes' energies get lost. They get emitted and some of it arrive at Earth.

Q1. How much energy is emitted from the collision?

$$k = 29\% \qquad \text{(Percentage of energy emitted)}$$

$$M = M_1 = M_2 = 5 \times 10^{31} kg \qquad \text{(Mass of the black holes before collision)}$$

$$c = 3 \times 10^{8} m/s \qquad \text{(Speed of Light)}$$

Energy Transmitted to Earth (Breakout Rooms)

Q2. How much energy is transmitted to Earth?

$$s = 1.23 \times 10^{25} m \qquad \text{(Separation between Earth and collision)}$$

$$R_E = 6 \times 10^{6} m \qquad \text{(Radius of the Earth)}$$

Hint: Think back to last time, how did we calculate the power received at Earth as a function of power radiated from the sun?

Energy Transmitted to Earth (Solutions)

Q1. How much of energy is emitted from the collision?

$$\Delta E = k(2M)c^2$$

Q2. How much energy is transmitted to Earth?

$$\begin{aligned} E_{Earth} &= \Delta E \times \frac{\pi R_E^2}{4\pi s^2} \\ &= 2kMc^2 \frac{\pi R_E^2}{4\pi s^2} \\ &= 1.18 \times 10^{11} J \end{aligned}$$

Energy Transmitted to Earth

We found that E_{earth} is massive – in fact, enough to power 1000 households for a day.

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Energy Transmitted to Earth

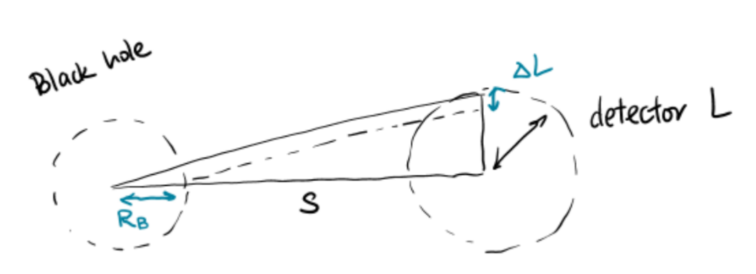
We found that E_{earth} is massive – in fact, enough to power 1000 households for a day.

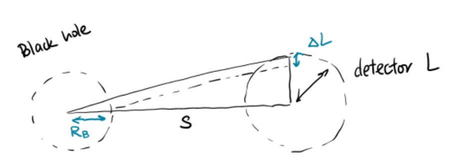
So... Why are blackhole collisions so hard to detect?

- We cannot learn much from the energy, nothing about its origin, could be black holes, could be anything...
- Scientists at LIGO instead study the **distortions** and its **frequency** to the experiment apparatus due to gravitational waves.

Calculating how much the actual gravitational waves distorted the LIGO apparatus is hard...
However, we can estimate it using the fact that "strain" is conserved:

$$\frac{R_B}{s} = \frac{\Delta L}{L}$$





Using the relation to find the distortion (ΔL) on LIGO apparatus:

$$\frac{R_B}{s} = \frac{\Delta L}{L} \qquad R_B = 2\frac{GM}{c^2}$$

$$G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$$ (Gravitational constant)

$$M = M_1 = M_2 = 5 \times 10^{31} kg$$ (Mass of the black holes before collision)

$$c = 3 \times 10^8 m/s$$ (Speed of Light)

$$s = 1.23 \times 10^{25} m$$ (Separation between Earth and collision)

$$L = 4 \times 10^5 m$$ (Size of the LIGO Apparatus)

With this is very elementary estimation, we found:

$$\Delta L = 1.2 \times 10^{-15} m$$
$$h \approx \frac{\Delta L}{L} \approx 10^{-21}$$

Using simple geometry, we are correct up to the same order of magnitude! However, the ΔL we can measure here is very small, about 10^{-5} the size of the atom... Can we do better?

Distortion (Solution)

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- h is dictated by the phenomenon we try to measure, which is hard to change.
- Even for small ΔL, we can get good results with better ($\approx$ more expensive) detectors.
- L is how big our experiment is, so if we just make it larger... Turns out this is hard!

Experiment Size (L) Considerations

Turns out LIGO is already tried to enlarge L!

- The size of LIGO is 4km. In the previous calculation we used $4 \times 10^5 m$, the extra factor of 100 is because the laser beam is reflected 100 times, effectively increasing L.

However, increasing L comes with its real life concerns. Any ideas?

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However, increasing L comes with its real life concerns. Any ideas?

- Cost of construction and maintenance
- Environmental noise
- Takes longer to vacuum
- Effects of curvature of the Earth
- The laser requires a certain operating frequency, which is closely related to L.

LIGO uses laser interferometers to detect gravitational waves, which has its best reponse when the wavelength of the laser matches that of the gravitational wave. As well, the wavelength of the laser should match the length of the experimental apparatus, so:

Transfer Frequency

$$L = \lambda_{GWave} = \lambda_{laser}$$
$$L = \frac{c}{2\pi f_{GWave}} = \frac{c}{2\pi f_{laser}}$$

We can use Keplar's law to approximate the frequency of the gravitational waves, which we had earlier modelled as two rotating black holes.

Keplar's Third Law

The square of a planet's orbit period is proportional to the cube of the radius of the orbit.

$$T^2 = (\frac{4\pi^2}{GM})R^3$$

Exercise:

1. Show Keplar's third law using force equilibrium between centripetal force and gravitational force.
2. The trajectory for a planet in orbit is an ellipse, how could we modify the statement above?

Laser Frequency and Detector Size (Breakout Room)

Recall:

$$\frac{c}{2\pi f_{GWave}} = \frac{c}{2\pi f_{laser}}$$
$$T^2_{GWave} = (\frac{4\pi^2}{GM})R^3_B$$

Using the two relations above, find the desired frequency for LIGO interferometers.

$$c = 3 \times 10^8 m/s \quad \text{(Speed of Light)}$$
$$G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \quad \text{(Gravitational constant)}$$
$$M = 5 \times 10^{31} kg \quad \text{(Mass of orbiting black hole)}$$
$$R_B = 74000m \quad \text{(Radius of the black hole, pre-collision)}$$

Laser Frequency and Detector Size (Solution)

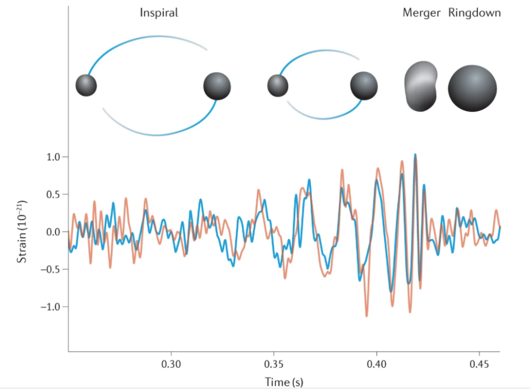
The transfer frequency of LIGO detector using our models would be

$$T_{GWave} = 2\pi\sqrt{\frac{R_B^3}{GM}}$$

$$f_{laser} = f_{GWave} = \frac{1}{T_{GWave}}$$

$$f_{laser} = \frac{1}{2\pi}\sqrt{\frac{GM}{R_B^3}} \approx 456 Hz$$

Another Look at the Ligo Data



Focusing on the region where the two black holes merged (near $t = 0.4s$), we see that:

$$\text{Strain} = \frac{\Delta L}{L} \approx 10^{-21}$$

$$\text{Frequency} = \frac{1}{T} = \frac{1}{0.01s} \approx 100Hz$$

Laser Frequency and Detector Size

Turns out we are off by a factor of 4! The actual frequencies of the Gravitational waves:

$$f_{laser} \approx 100Hz \sim 150Hz$$

We can then estimate the length of the experiment needed:

$$L = \frac{1}{2\pi f_{laser}} \approx 4 \times 10^5 m$$

This is quite closer to the size of LIGO. Nicely done future particle physicists :)