The Physics of Rocket Stages !

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UBC Physics Circle
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Overview

Introductions

Using energy methods to determine velocities required for space travel

Using momentum conservation to calculate rocket velocities

The simple physics of rocket engines

Student-built 3 km rocket

Air drag by dimensional analysis

Student-built 10 km rockets

Student-built 100 km rocket?

Orbital velocity and multi-stage rockets

Extra Credit (if I have time)

About Me

Grew up in Chicago suburbs, near Fermilab (particle accelerator lab)
B.Sc. in physics from MIT (bubble chamber data thesis)
Ph.D. in physics from MIT (neutrino experiment at Fermilab)

Post-doctoral job at Stanford Linear Accelerator Centre
(e$^+$e$^-$ linear collider accelerator, Mark II detector for Z^{0}s)

CERN for a few years (ALEPH detector for Z^{0}s at LEP, before LHC)

Back to SLAC (PEP-II B-Factory accelerator)

At UBC since 1999 (BaBar detector at SLAC B-Factory for CP-violation)

These days mostly I help UBC Rocket

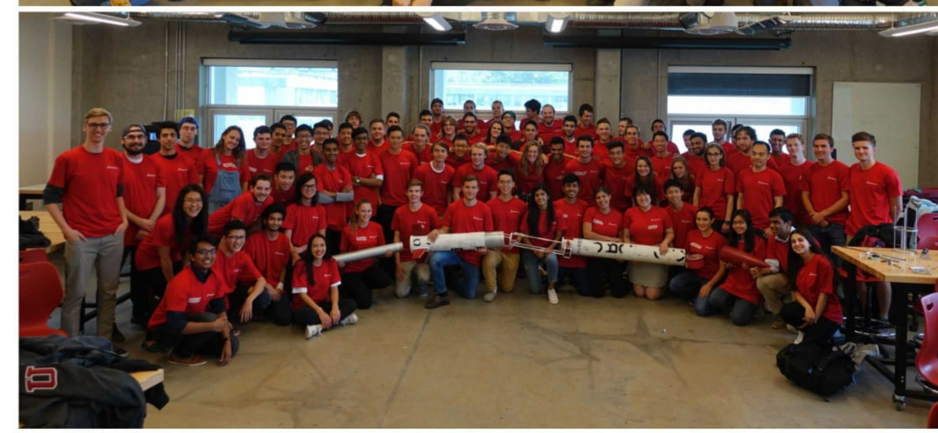
About UBC Rocket

Started in 2016.

Won first place in 3 km commercial engine division at International Rocket Engineering Competition at Spaceport America in New Mexico (~150 teams)

At Launch Canada 2023 in Timmins, Ontario, won first place in 10 km advanced competition, and overall Launch Canada trophy (12 teams).

Now the largest engineering student design team at UBC, with about 100 members.



Using Energy to Find Velocity Required

Kinetic energy is $K.E. = \frac{1}{2}mv^2$.

Potential energy in Earth-surface gravity is $P.E. = mgh$.

Set them equal: $\frac{1}{2}mv^2 = mgh$.

Solve for $v = \sqrt{2gh}$.

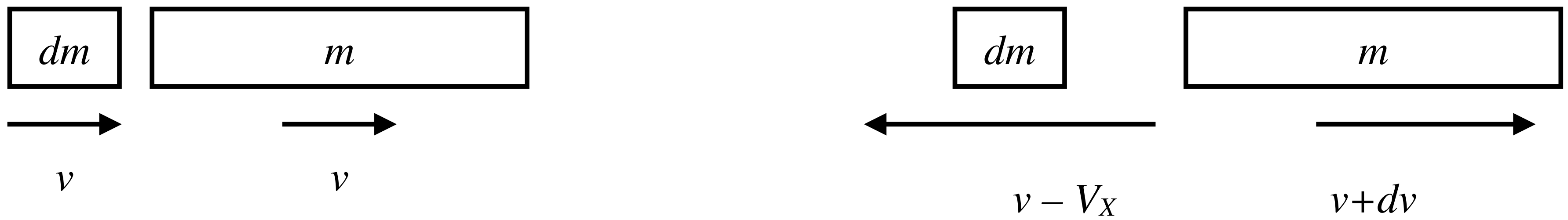
To get 3 km high (3000 m), $v = \sqrt{2 \cdot 9.81 \cdot 3000} = 243\,\text{m/s}$.

To get 10 km high, $v = \sqrt{2 \cdot 9.81 \cdot 10000} = 443\,\text{m/s}$. (Speed of sound is 343 m/s).

To get 100 km high (the Karman line), $v = \sqrt{2 \cdot 9.81 \cdot 10^5} = 1401\,\text{m/s}$

Rocket Momentum Relation

A rocket engine works by ejecting mass dm at velocity V_X relative to the engine.



Masses dm and m have initial velocity v, and total momentum $(m+dm)\cdot v$

After dm is ejected with relative velocity V_X, the velocity of m increases by dv.

The total momentum is then $m\cdot(v+dv)+dm\cdot(v-V_X)$.

Momentum is conserved, so $(m+dm)\cdot v = m\cdot(v+dv)+dm\cdot(v-V_X)$.

Do the cancellations to get $0 = m\cdot dv - V_X\cdot dm \;\rightarrow\; \boxed{m\cdot dv = V_X\cdot dm}$

Rocket Thrust Equation

If we divide both sides of $m \cdot dv = V_X \cdot dm$ by dt,

we get $m\dfrac{dv}{dt} = V_X \dfrac{dm}{dt}$.

But $m\dfrac{dv}{dt} = ma$ and Newton says $F = ma$, so we have $\boxed{F = V_X \dfrac{dm}{dt}}$

So the thrust force F is the exhaust velocity V_X times the mass-flow $\dfrac{dm}{dt}$.

Tsiolkovsky's Rocket Equation

We can rearrange $m \cdot dv = V_X \cdot dm \rightarrow \frac{dv}{V_X} = \frac{dm}{m}$ then integrate both sides.

But we add a minus sign, because m goes down by dm, not up by dm:

$$\int_{v=v1}^{v=v2} \frac{dv}{V_X} = \int_{m=m1}^{m=m2} \frac{-dm}{m}$$

$$\frac{v_2 - v_1}{V_X} = \left[-\ln m\right]_{m=m1}^{m=m2} = -\ln m_2 - \left(-\ln m_1\right) = \ln m_1 - \ln m_2 = \ln \frac{m_1}{m_2}$$

$$\boxed{v_2 - v_1 = V_X \cdot \ln \frac{m_1}{m_2}}$$

The velocity change is the exhaust velocity V_X
times the natural log of the initial mass m_1 over the final mass m_2.

To get a high velocity, we want high V_X and high initial mass (lots of fuel).

Alternative Forms

If $v_1 = 0$, then $v_2 = V_X \cdot \ln \frac{m_1}{m_2}$, so $\frac{v_2}{V_X} = \ln \frac{m_1}{m_2}$

Exponentiate both sides to get $\boxed{e^{\frac{v_2}{V_X}} = \frac{m_1}{m_2}}$

If $m_2 = m_{\text{burnout}}$ and $m_1 = m_{\text{burnout}} + m_{\text{fuel}}$

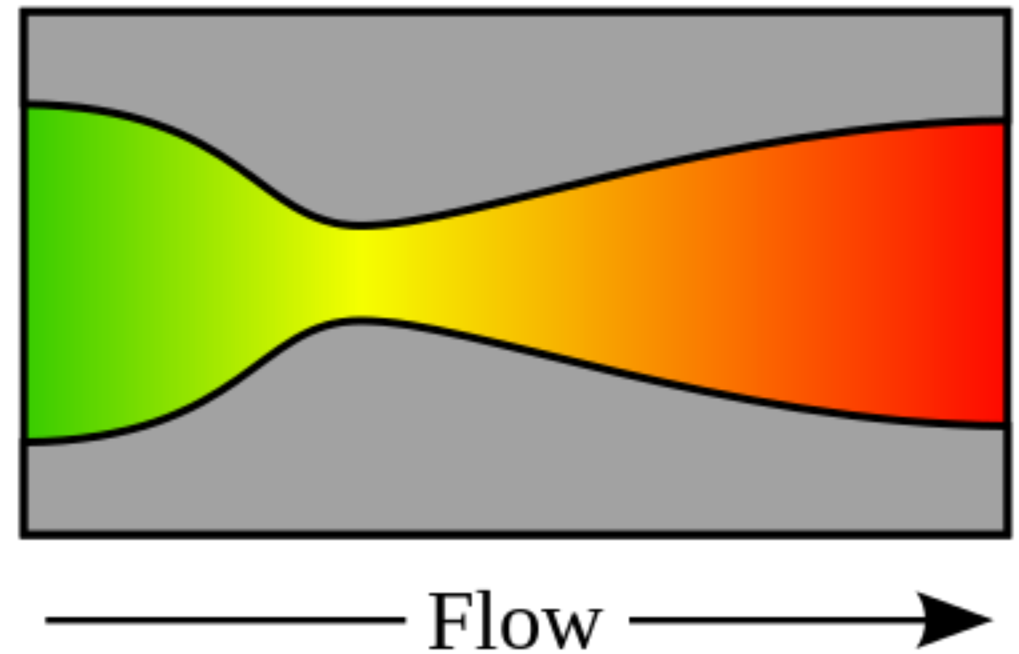
then $\frac{m_1}{m_2} = \frac{m_{\text{burnout}} + m_{\text{fuel}}}{m_{\text{burnout}}} = 1 + \frac{m_{\text{fuel}}}{m_{\text{burnout}}}$

Plug that in: $e^{\frac{v_2}{V_X}} = 1 + \frac{m_{\text{fuel}}}{m_{\text{burnout}}}$.

Then $e^{\frac{v_2}{V_X}} - 1 = \frac{m_{\text{fuel}}}{m_{\text{burnout}}}$ or $\boxed{m_{\text{burnout}} = \frac{m_{\text{fuel}}}{e^{\frac{v_2}{V_X}} - 1}}$

Rocket Engine Design

Most rocket engines turn chemical (or nuclear) energy into gas thermal energy, then let the gas expand, changing the thermal energy into gas kinetic energy.



The flow can be predicted by using the adiabatic gas law $TV^{\gamma-1} = \text{constant}$, increasing velocity as the temperature decreases, and conserved mass-flow.

The radius decreases until the gas flow reaches the speed of sound (at the throat temperature), then must increase to further expand the gas. The color here is flow velocity, not temperature.

In the sub-sonic flow upstream of the throat, the convergence can be rapid.
In the super-sonic flow downstream, the divergence must be gradual,
or the flow separates from the wall and some energy is wasted.

The expanded gas carries momentum, and the rocket gains opposite momentum.

Exhaust Velocity Physics

The average kinetic energy of molecules in a gas is $\frac{1}{2}m\left\langle v^2\right\rangle = \frac{3}{2}k_B T$

where m is the mass of a molecule, $\left\langle v^2\right\rangle$ is the average squared-velocity,

k_B is Boltzman's Constant, and T is the temperature.

So $\boxed{\sqrt{\left\langle v^2\right\rangle} = \sqrt{3k_B \frac{T}{m}}}$

It doesn't violate physics for a nozzle to make all the molecule velocities parallel, so that's the maximum possible exhaust velocity for a thermal rocket.

The nozzle would have to be infinitely long with infinite expansion to get to the theoretical maximum, but getting to a significant fraction of it is feasible.

So we want the maximum temperature, and the minimum molecular mass.

Nuclear Thermal Rocket

The gas with the lowest molecular weight is hydrogen.

A fission reactor reactor can heat hydrogen to about the same temperature as burning hydrogen with oxygen, but without molecular weight of the oxygen.

This has been tested on the ground, and produces exhaust velocities of 8.5-10 km/s, about twice that of a hydrogen-oxygen rocket.

Some fuel elements were damaged in tests, and fission products were emitted, but this could probably be solved. (They also intentionally melted one down…)

I doubt this would ever be used for launch from Earth,
but might be used to go from Earth orbit to other planets.

Hydrogen-Oxygen Rocket

The reaction is $2\mathrm{H}_2 + \mathrm{O}_2 \rightarrow 2\mathrm{H}_2\mathrm{O}$

You get the maximum temperature with 4 kg of H_2 per 32 kg of O_2 so the oxidizer/fuel mass ratio is 8. The product is H_2O with molecular weight of 18.

It's better to use about twice as much H_2, for an oxidizer/fuel ratio of 4. Basically, the chemical reaction heats the excess hydrogen, like a nuclear rocket.

The extra H_2 lowers the temperature, which is bad for exhaust velocity, but lowers the average molecular weight a lot, which more than compensates.

Actually, the temperature is so high that a lot of the exhaust is single hydrogen atoms rather than H_2, which lowers the average molecular weight. And there's a lot of OH as well, although that doesn't help as much.

Exhaust velocity is about 4.5 km/s.

Hydrocarbon-Oxygen Rocket

The reaction is $\mathrm{H}\left(CH_2\right)_n\mathrm{H} + \left(1.5n + 0.5\right)\mathrm{O}_2 \rightarrow nCO_2 + \left(n+1\right)H_2O$

For large n, the oxygen to fuel mass ratio for maximum temperature is 3.4.

But again, it's better to use excess fuel, which gets broken down by the heat to produce H_2 and even mono-atomic H, to lower the average molecular weight.

The traditional hydrocarbon fuel is basically kerosene ($n \sim 12$).
Best exhaust velocity is with oxygen/kerosene mass ratio of around 2.3.

Methane ($n = 1$) has a bit better exhaust velocity (and can be made on Mars).

Excess kerosene burns in the air outside the engine, making flame and soot.
(Hydrogen and methane engines have external burning, but it's almost invisible).

Typical exhaust velocity is about 3 km/s.

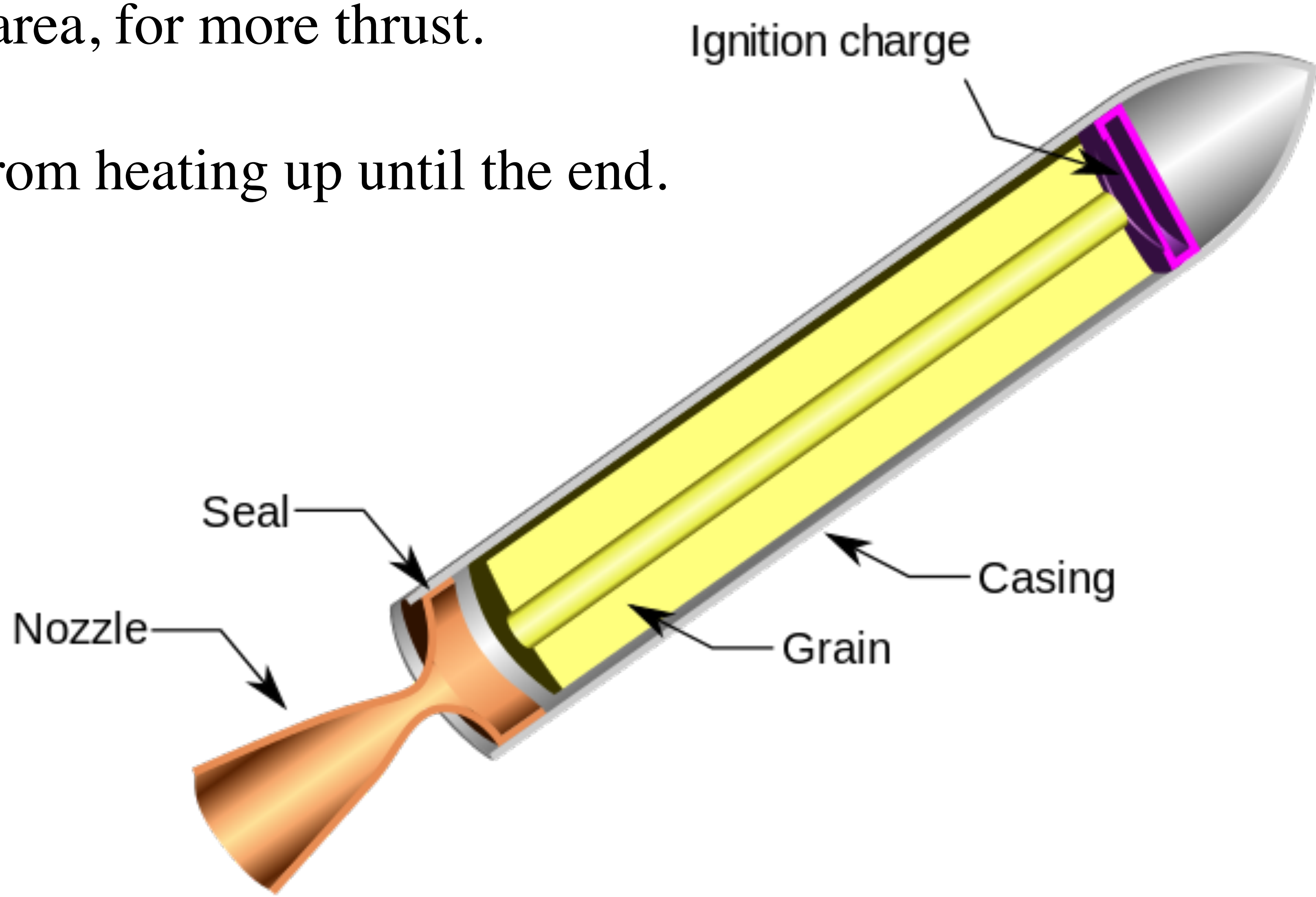
Solid-Fuel Rocket

Solid fuels are mixtures of an oxidizer and a fuel,
inside a strong thus fairly heavy case.

There's a hole through the fuel, so it burns from inside to outside.

That gives more surface area, for more thrust.

It also keeps the casing from heating up until the end.



Solid Fuel Exhaust Velocity

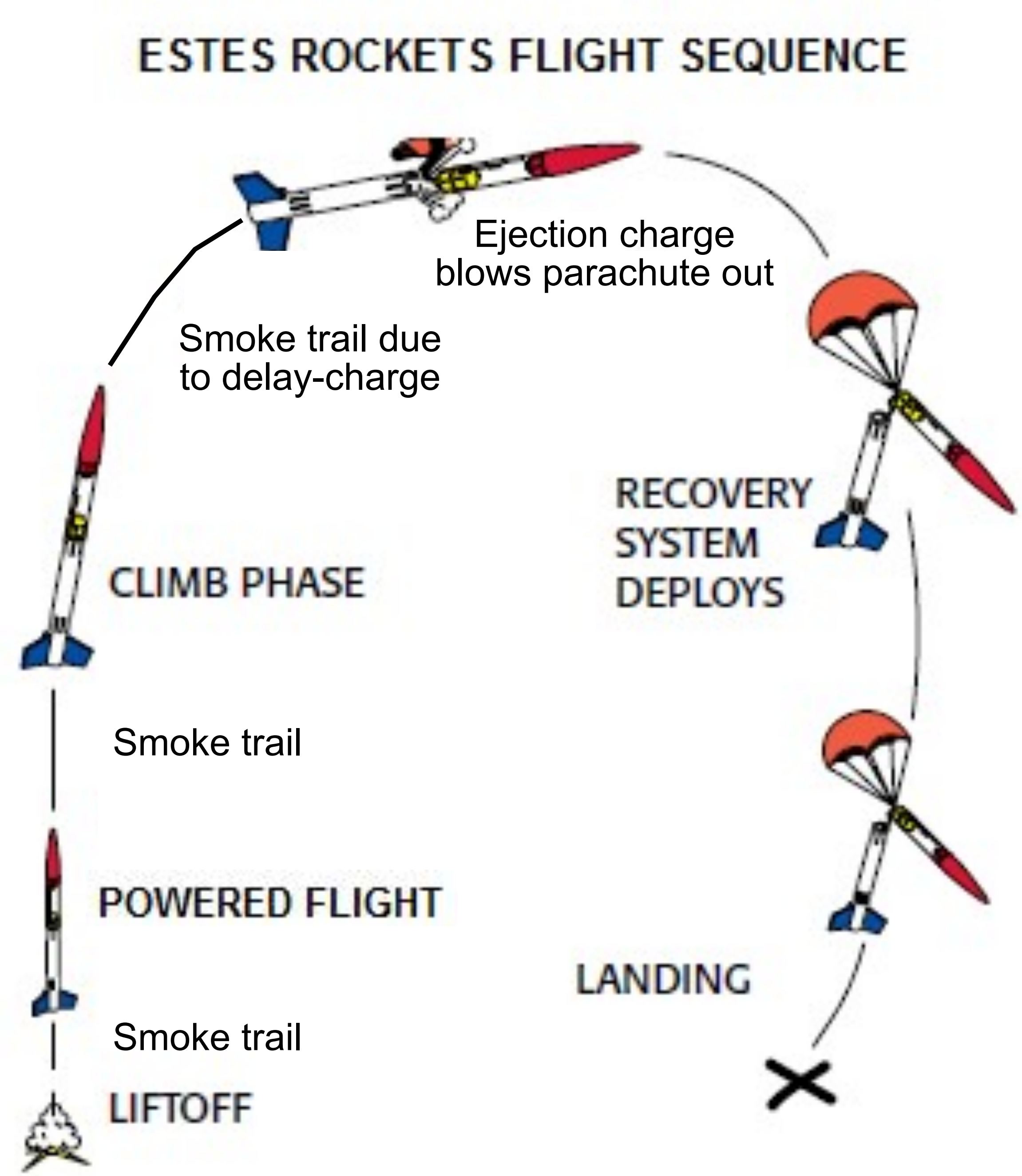
Small commercial hobby-rocket engines use black powder:
KNO_3 as the oxidizer, and charcoal as the fuel (sulphur is both).
Exhaust velocity is about 900 m/s.

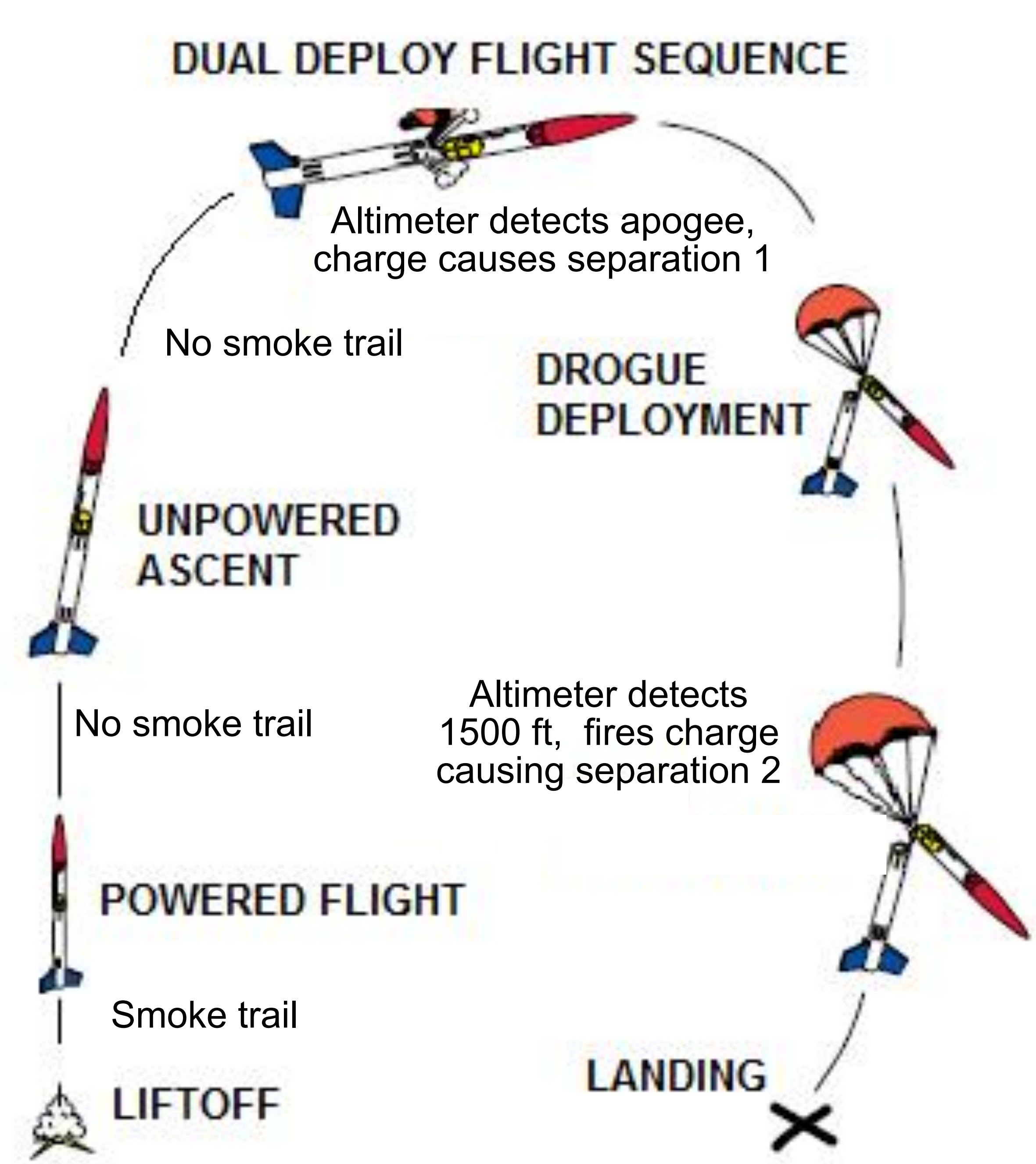
KNO_3 and sugar (or sorbitol) “rocket-candy” gives about 1100-1300 m/s.

Modern solid rocket motors (both professional, and large hobby motors)
use ammonium perchlorate ($NH_4 Cl O_4$) as oxidizer, powdered aluminum as fuel,
and plastic as both binder and fuel. Exhaust velocity is 2-2.5 km/s.

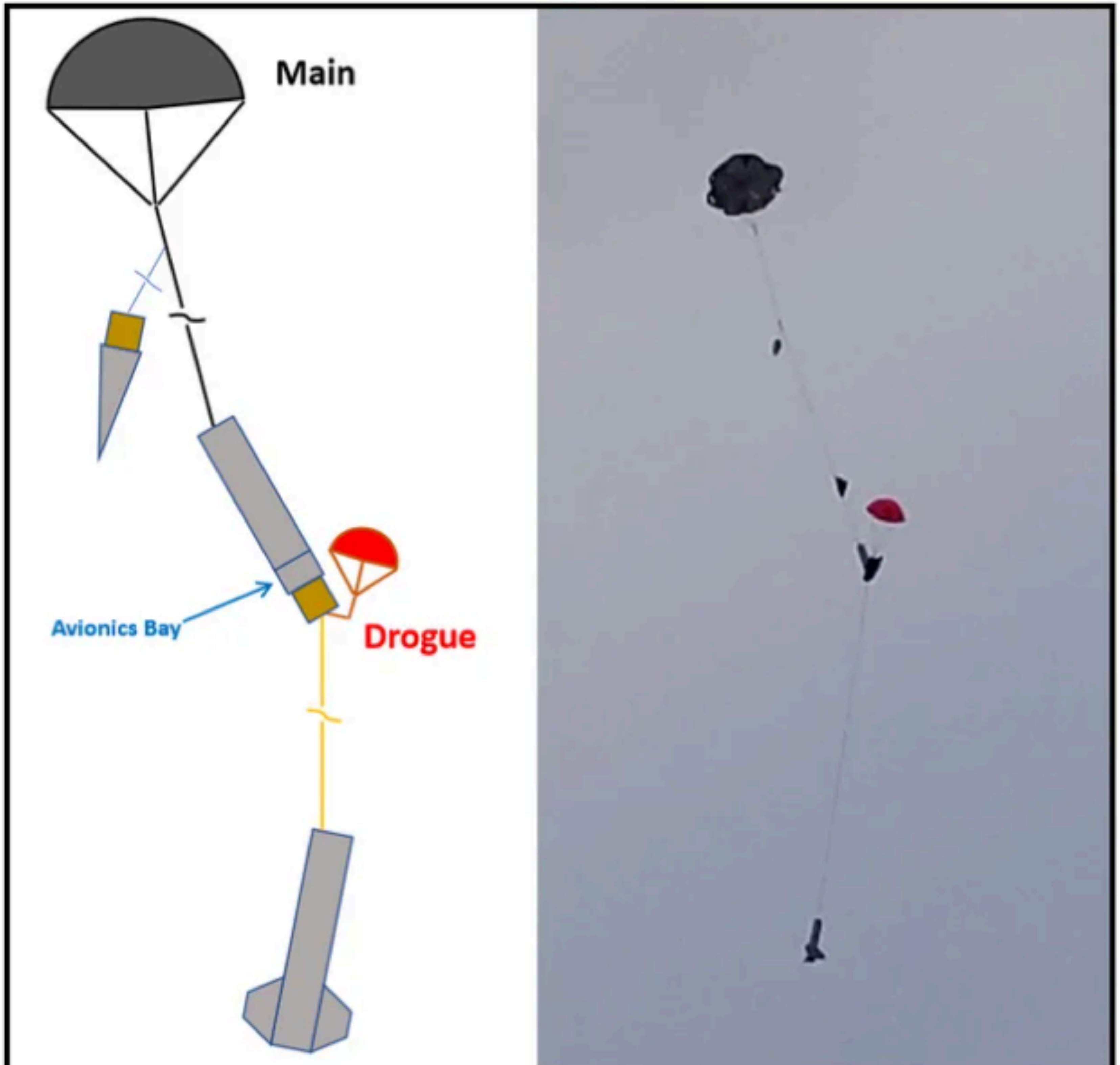
This is lower than liquid fuels because there is less chemical energy,
and more high-molecular-weight reaction products.

Student Rocket Recovery



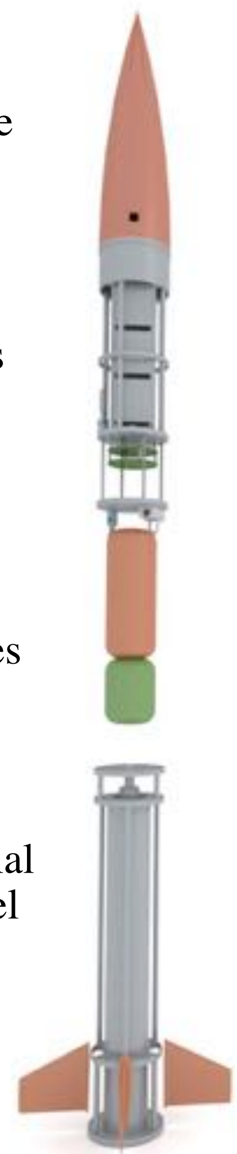


Dual Separation Configuration



3 KM Solid-Fuel Rocket

Cypress



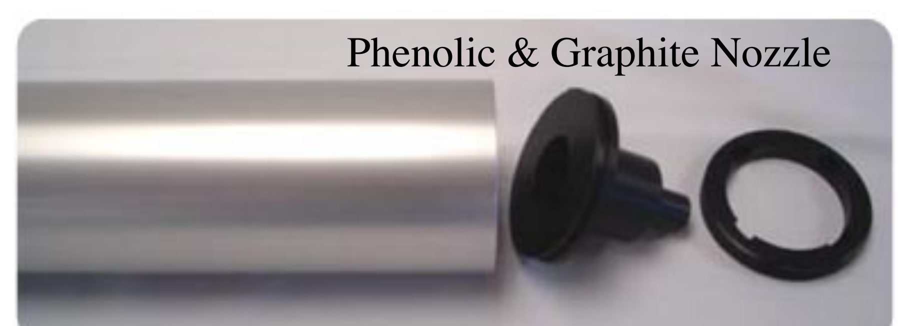
Nosecone

Payload
Avionics
Camera

Drogue
& Main
Parachutes

Commercial
Solid-Fuel
Motor

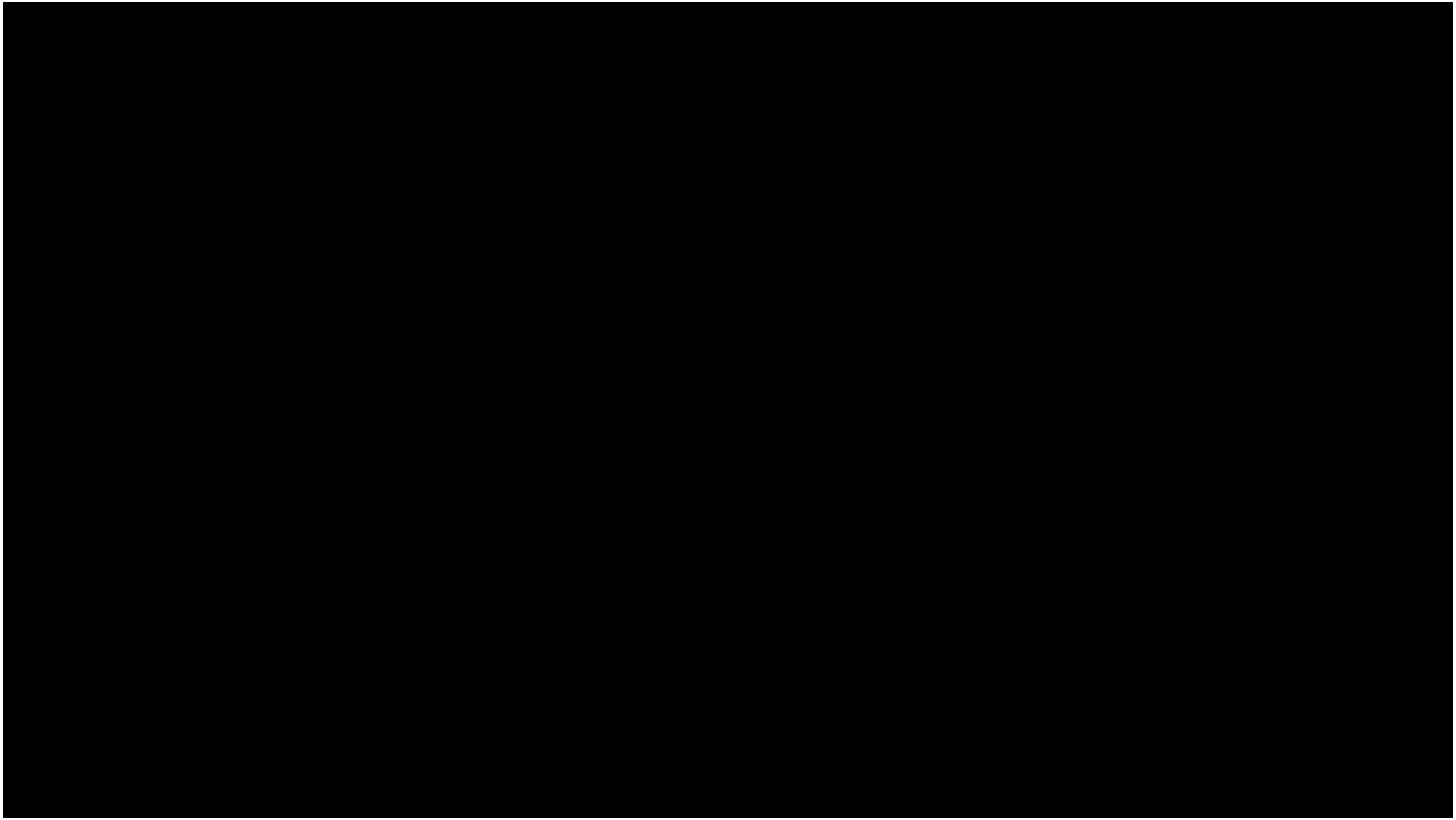
Fins



Phenolic & Graphite Nozzle

Body Separation Test

Parachute Test



Flight !

Cypress Analysis

The motor had 5 kg of solid fuel with exhaust velocity $V_X = 2000\ \text{m/s}$.

The velocity required for 3 km is 243 m/s

The alternative form of Tsiolkovsky is $m_{\text{burnout}} = \dfrac{m_{\text{fuel}}}{e^{\frac{v}{V_X}} - 1}$.

Plug in: $m_{\text{burnout}} = \dfrac{5\ \text{kg}}{e^{\frac{243}{2000}} - 1} = \dfrac{5\ \text{kg}}{1.13 - 1} = 38.7\ \text{kg}$.

So the 5 kg of fuel could boost a 38.7 kg rocket to 3 km.

The empty mass of the engine casing alone is 3.6 kg.

It's pretty easy to keep the rocket plus payload below 38.7 kg.

Air Drag by Dimensional Analysis

Dimensionally, energy = work = force times distance = $F \cdot L$.

Dimensionally, volume = area times length = $A \cdot L$.

Energy per volume = $\dfrac{F \cdot L}{A \cdot L} = \dfrac{F}{A}$ which has dimensions of pressure.

Energy $= \dfrac{1}{2} m v^2$. Energy per volume $= \dfrac{\frac{1}{2} m v^2}{\text{volume}} = \dfrac{1}{2} \dfrac{m}{\text{volume}} v^2 = \dfrac{1}{2} \rho v^2$

where $\boxed{\rho = \dfrac{\text{mass}}{\text{volume}}.}$

So $\boxed{Q = \dfrac{1}{2} \rho v^2}$ has the dimensions of pressure. It's called the <u>dynamic pressure</u>.

Dimensionally, force = pressure times area.

Dimensionally, drag force should be $Q \cdot A \cdot C_d = \boxed{F_{\text{drag}} = \dfrac{1}{2} \rho v^2 \cdot A \cdot C_d}$

where C_d is a dimensionless drag-coefficient.

Cypress Air Drag Analysis

C_d depends on the shape, and can also depend on the velocity.
For a typical subsonic rocket, it's around 0.75.

The density of air at sea-level is about 1.3 kg/m^3.
Cypress was 6.5 inches = 16.5 cm diameter, for $A = 0.0214\ \mathrm{m}^2$.

At v = 243 m/s, drag force is $F_d = \frac{1}{2} \cdot 1.3 \cdot 243^2 \cdot 0.0214 \cdot 0.75 = 615.5\ \mathrm{N}$.
That's equivalent to a weight of 62.8 kg, much higher than the burnout weight.

So the weight must be much less than 38.7 kg to get to 3 km on 5 kg of fuel.

There are computer programs that calculate drag vs velocity from geometry, and the variation of density with height, and you can give them detailed thrust vs time curves, to predict the peak altitude of the rocket.

UBC Rocket added ballast to Cypress until the predicted altitude was 3 km.

10 km Rocket Analysis

The velocity required for 10 km is 443 m/s (ignoring air drag).
The largest available commercial solid fuel engine has 9 kg of fuel.

Plug in: $m_{\text{burnout}} = \dfrac{9\text{ kg}}{e^{\frac{443}{2000}} - 1} = \dfrac{9\text{ kg}}{1.248 - 1} = 36.3\,\text{kg}$.

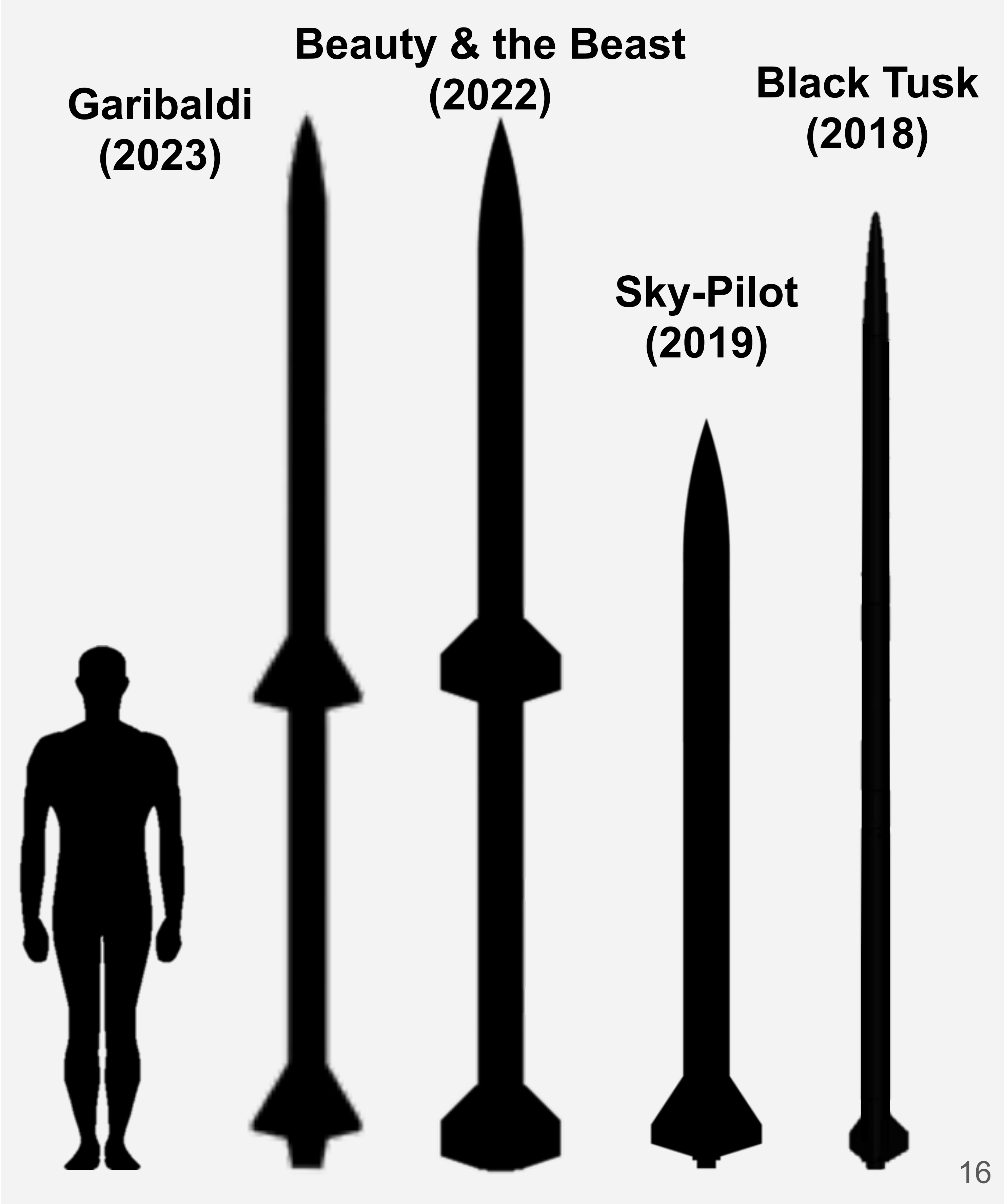
So the 9 kg of fuel could boost a 36.3 kg rocket to 10 km (ignoring air drag).

But, the air drag at 443 m/s is $F_d = \dfrac{1}{2} \cdot 1.3 \cdot 443^2 \cdot 0.0214 \cdot 0.75 = 2047\text{ N}$.
That's equivalent to a weight of 209 kg.

So air drag is a <u>much</u> larger issue for a 10 km rocket than a 3 km rocket.

You must reduce the air drag, greatly reduce the rocket mass, or use 2 engines.

UBC 10 km Rocket Designs



Black Tusk

Use longest motor available (9 kg fuel).
Reduce diameter from 6.5 inches to 4 inches (diameter of the motor).
Reduce mass by making carbon-fibre tubes with thinner walls. Reduce payload.

Black Tusk

Use longest motor available (9 kg fuel).
Reduce diameter from 6.5 inches to 4 inches (diameter of the motor).
Reduce mass by making carbon-fibre tubes with thinner walls. Reduce payload.
Broke up in flight slightly after going supersonic.

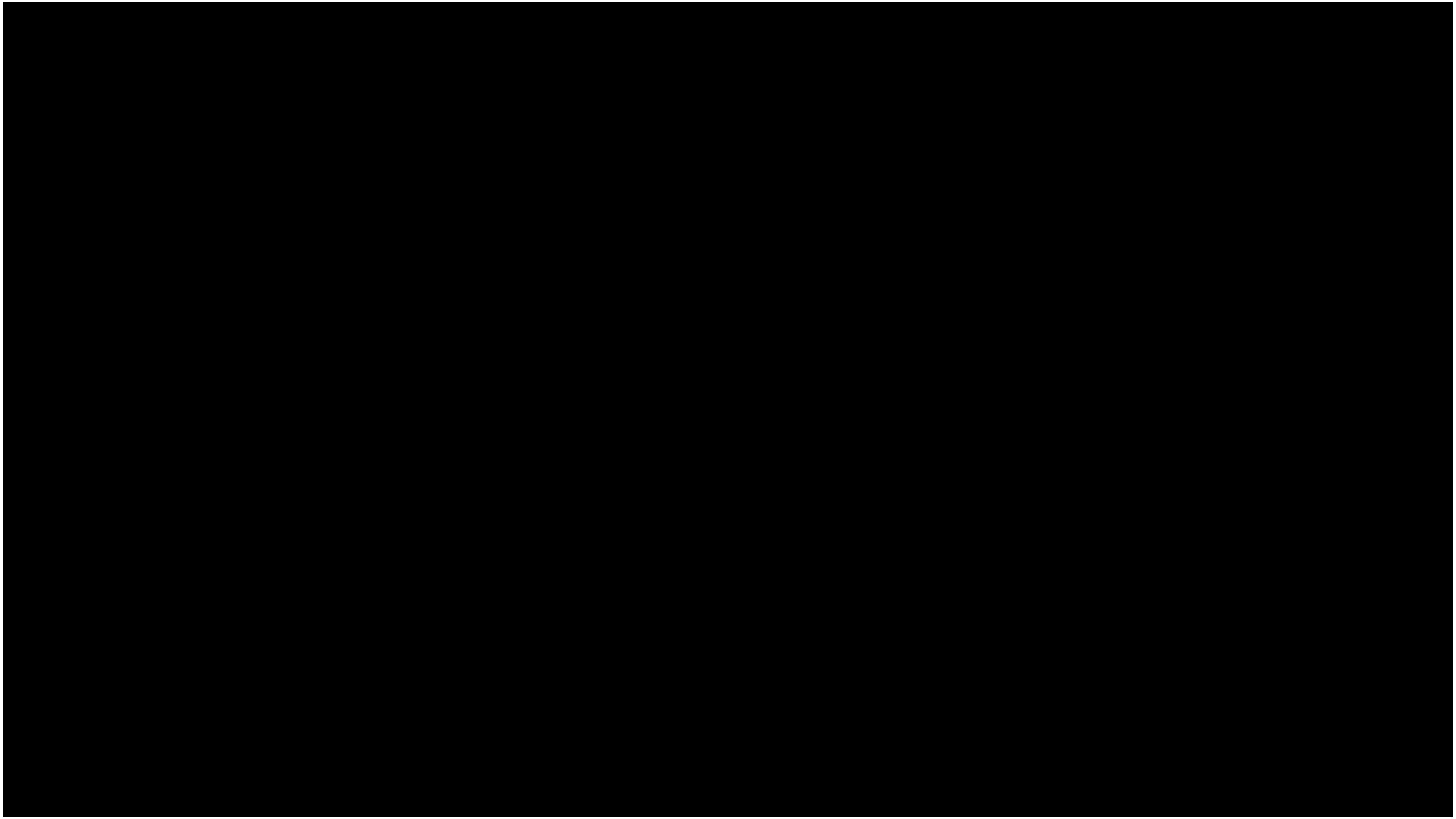
Sky Pilot

6.5 inch diameter, “Boat-tail” to reduce base-drag. Nice launch, but didn’t get to 10 km. Drogue chute deployed, but main did not, so some landing damage.

Tantalus

Use two engines! First stage uses maximum size engine, stays attached for a few seconds so its kinetic energy helps punch through air drag. Second stage should ignite in flight to increase velocity to get to 10 km. But in both launch attempts, it was so far from vertical at ignition time that electronics vetoed ignition.

Beauty & the Beast (at Launch Canada)

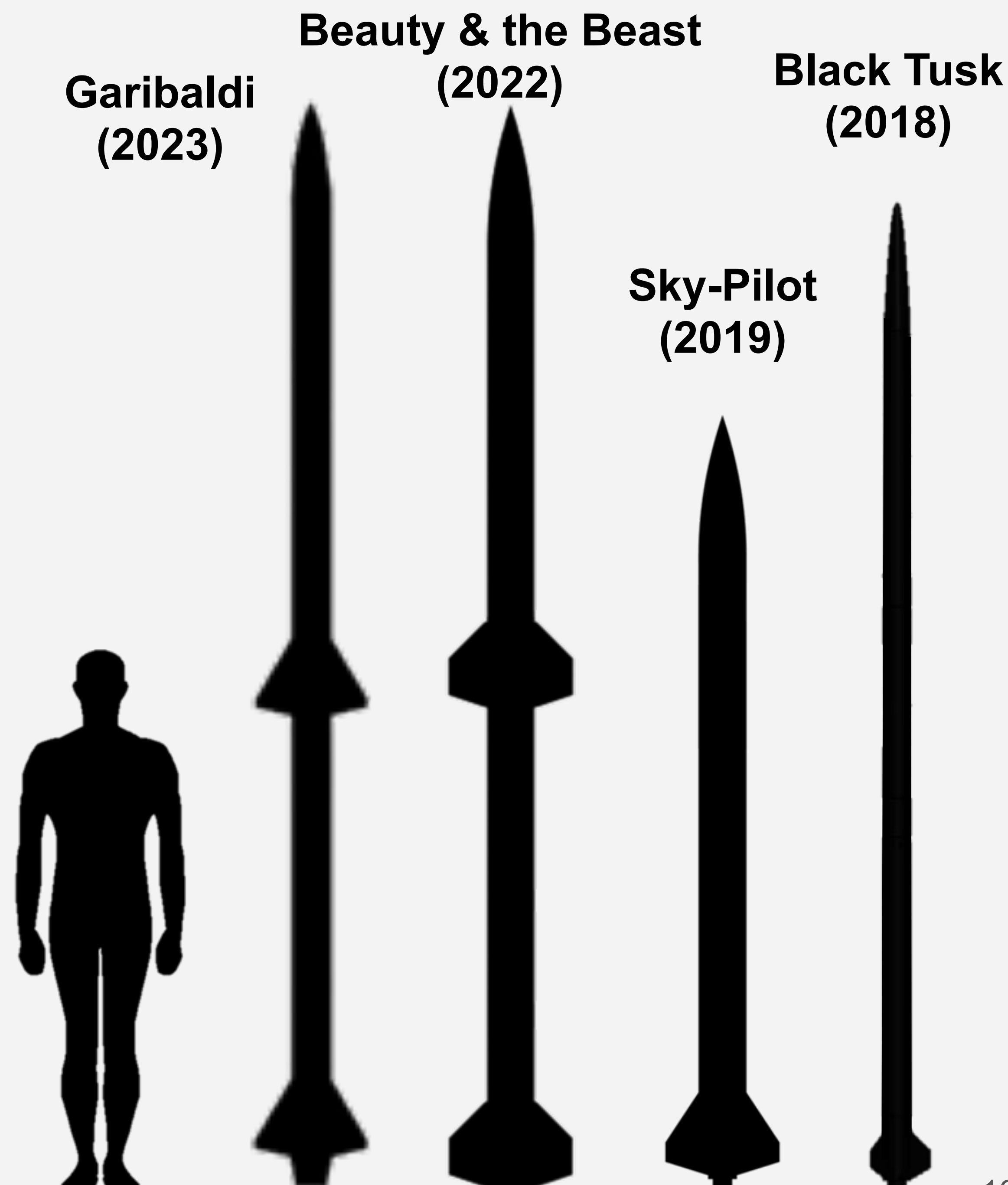


It worked! And UBC won, both the “advanced” flight catagory, and the overall trophy (Bonus liquid-fuel hotfire by Concordia, but after judging deadline).

2023-4 Project: Garibaldi

Overview

Body Diameter	**156.4 mm**
Total Length	**433 cm**
Loaded Mass	**52.4 kg**
Dry Mass (no fuel)	**38.5 kg**
Total Motor Impulse	**27,891 Ns**
1st stage average thrust	**5600 N**
2nd stage average thrust	**3180 N**



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Body diameter reduced from 6.5 to 6.0 inches, plus “boat-tail.”
13.9 kg total fuel !

100 km Rocket Analysis

The velocity required for 100 km is 1400 m/s (ignoring air drag). Mach 4!

Use alternative form $\dfrac{m_{\text{fuel}}}{m_{\text{burnout}}} = e^{\frac{v_2}{V_X}} - 1 = e^{\frac{1400}{2000}} - 1 = 1.014$

The fuel mass needs to be a bit larger than the burnout mass.

The air drag at 1400 m/s at sea level for 30 cm diameter is

$$F_d = \frac{1}{2}\rho v^2 \cdot \frac{\pi D^2}{4} \cdot C_d = \left(\frac{1}{2} \cdot 1.3 \cdot 1400^2\right) \cdot \left(\frac{\pi \cdot 0.3^2}{4}\right) \cdot 0.75 = 67{,}540 \text{ N}.$$

That’s equivalent to a weight of 6890 kg.

A military solid-fuel rocket can get to 100 km, but students can’t buy one.

So build a liquid-fuel engine, and big fuel tanks. And burn the fuel slowly, so the high speed happens at high altitude where the air density is lower.

Whistler-Blackcomb Project

Design and build a rocket to get to 100 km. For a while, there was a competition with a million dollar (US) prize, and UBC was a leader. But it had a deadline, and no one met it.

Has built several generations of liquid-fuel engines with design thrust similar to competition solid fuel engines.

Built a test stand for “cold-flow” tests (using water).

Built a test-stand for “hot-fires” burning kerosene and liquid oxygen.

Tested an uncooled “heat-sink” engine, a “film-cooled” engine, an “ablative” engine, and a lower-mass “heat-sink” engine. Still ongoing.

Hope to have a liquid rocket flying next year.
Less than 100 km, but hopefully a student altitude record for liquid-fuel.

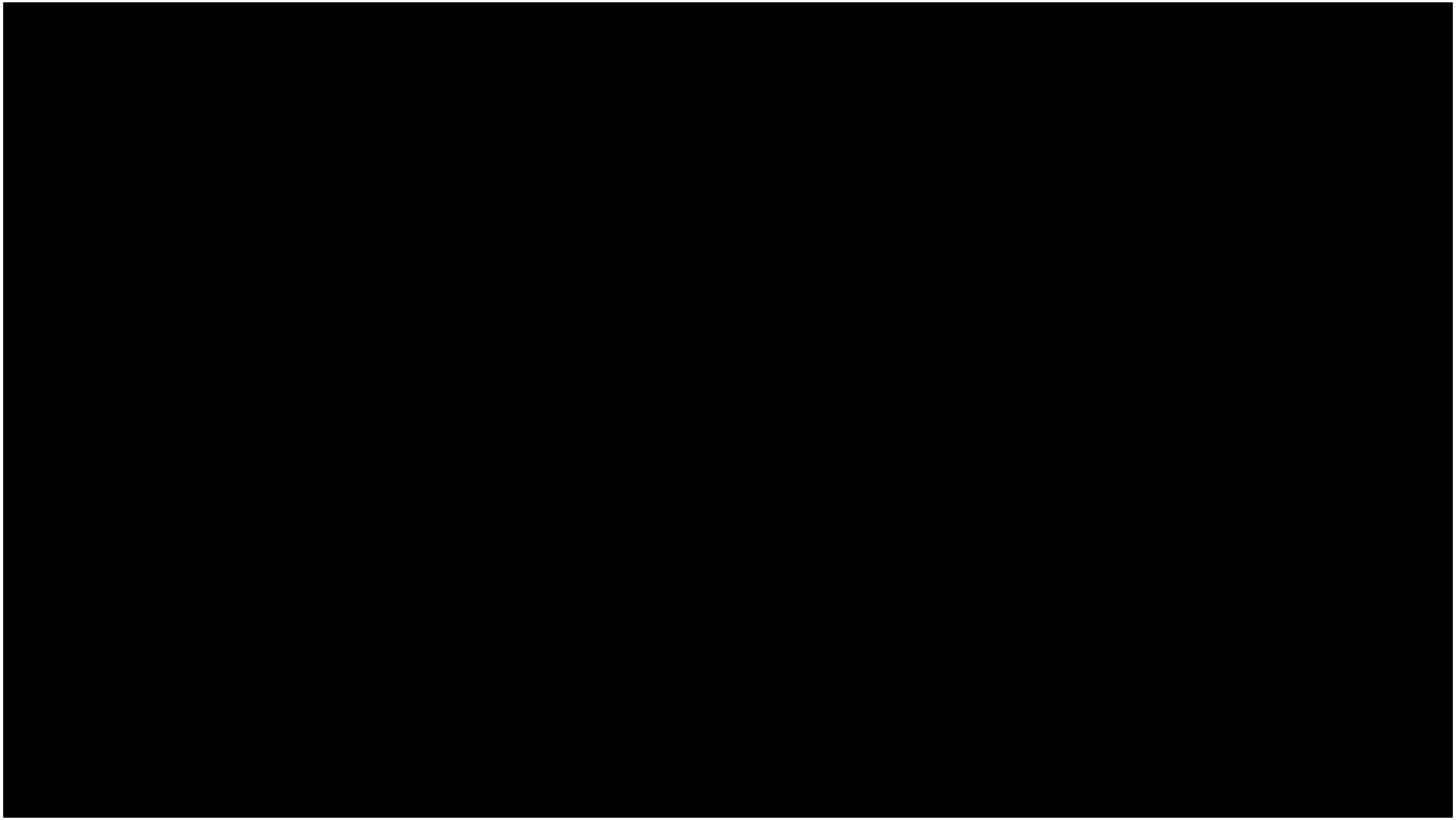
Cold-Flow

Imagine if that was kerosene and liquid oxygen….

Heat-Sink Hot-Fire

Actually, you don’t have to imagine….

Film-Cooled Hot-Fire



There was a second test the same day. The nozzle blew off…

Ablative Engine Hot-Fire

The ablative was rings of MDF (medium density fibreboard) !
Which held up better than the welds on the steel case around them…

Orbital Velocity

The gravity force is $F = \frac{GMm}{r^2}$. For a circular orbit, the acceleration is $a = \frac{v^2}{r}$.

Plug into $F = ma$ and solve $\frac{GMm}{r^2} = m\frac{v^2}{a}$ for $v_{\text{orbit}}(r) = \sqrt{\frac{GM}{r}}$.

I don't remember the value of G, or the mass of the Earth. But I do know a trick.

The meter was defined so the distance from pole to pole is 20,000 km,
so $\pi R_E = 20\times10^6$. Then $R_E = 20\times10^6/\pi = 6.366\times10^6 \text{ m} = 6366 \text{ km}$.

At Earth's surface, $mg = \frac{GMm}{R_E^2}$, so $GM = gR_E^2$.

Plug in $GM = gR_E^2$ to get $v_{\text{orbit}}(r) = \sqrt{\frac{gR_E^2}{r}}$

Orbital Velocity 2

An orbit at R_E has $v = \sqrt{\dfrac{gR_E^2}{R_E}} = \sqrt{gR_E} = \sqrt{9.81 \cdot 6.366 \times 10^6} = 7.903$ km/s

The International Space Station orbit is 400 km above the surface,

so $v = \sqrt{\dfrac{gR_E^2}{R_E + 400\text{ km}}} = \sqrt{\dfrac{9.81 \cdot \left(6.366 \times 10^6\right)^2}{6.766 \times 10^6}} = 7.665$ km/s

That's a bit misleading, because in addition to the kinetic energy of the orbit, you also need to increase the potential energy by 400 km. So a rocket needs to work harder to get to ISS than to get to a minimum-altitude orbit.

There is also loss due to air drag, and ascending above the air-drag.
A common rule of thumb is to add about 1 km/s for those loses.

So let's call the velocity requirement 9 km/s.

Orbital Rocket Analysis

The velocity required for 9000 m/s (including air drag). Mach 26!

Assume $V_X = 3000$ m/s for kerosene. Use the form $\dfrac{m_1}{m_2} = e^{\frac{v_2}{V_X}} = e^{\frac{9000}{3000}} = 20.1$

The launch mass needs to be 20 times the burnout mass.
So the payload can only be 5% of the launch mass. The rest must be fuel.

But it's worse than that, because the rocket structure mass isn't negligible.

If the fuel is hydrogen with $V_X = 4500$ m/s, $\dfrac{m_1}{m_2} = e^{\frac{v_2}{V_X}} = e^{\frac{9000}{4500}} = 7.38$.

That would allow a payload of 13.5% of the launch mass.

But hydrogen is very low density, so the tanks are big, and must be insulated.
So the extra tank mass for hydrogen cancels a lot of the advantage.

Structural Mass

The retiring Delta IV holds 200 tons of LH2 and LOX and weighs 26.8 tons empty, so 11.8% of the total mass is structure. That would leave a tiny payload.

But at sea level, Delta IV only has $V_X = 3.6$ km/s, so the final/total mass needs to be 8.4%. So it can't get a payload to orbit.

The SLS core stage holds 895 tons of LH2 and LOX and weighs 85 tons empty so 8.7% of the total mass is structure. But the hydrogen engines can't lift it. Adding extra engines adds to the mass, and the already obscene cost. And it would have the same lower V_X at sea level as the Delta IV. Doesn't work.

The retiring Atlas V first stage holds 284 tons of LOX and kerosene, and is 21 tons empty including engines, so 6.9% of the mass is tanks and engines.

The Falcon 9 first stage holds 396 tons of LOX and kerosene and has a mass of 25.6 tons empty including engines, so 6.1% of the mass is tanks and engines.

Multi-Stage Rockets

Tsiolkovsky not only derived the rocket equation, he noted that fuel tank mass would limit the velocity that a rocket could achieve.

But he also invented the solution: put a smaller rocket on top of a bigger rocket.

When the bigger rocket, with its big heavy tank, is out of fuel, jettison it, and ignite the smaller rocket. It starts with the burnout velocity of the big rocket, and adds it own velocity.

So each stage only needs to provide a fraction of the target velocity.

With 2 stages, a hydrocarbon rocket can get a reasonable payload into orbit.

Even solid fuel can get a reasonable payload to orbit with 3 stages, although often 4 stages are used.

Two-Stage Hydrocarbon Rocket

The velocity goal is 9000 m/s, with $V_X = 3000$ m/s.

But each stage only needs to provide half of the velocity, or 4500 m/s.

That gives $\dfrac{m_1}{m_2} = e^{\frac{v_2}{V_X}} = e^{\frac{4500}{3000}} = 4.48$.

So the non-fuel mass can be 1 / 4.48 = 22.3% of the launch mass.

Even if the non-fuel mass is 12%, twice as bad as Atlas V or Falcon 9, the second stage mass can be 10% of the launch mass.

The second stage could also have 22.3% of its mass as non-fuel, which might be 12% of its initial mass, which leaves 10% for payload.

The payload is 1% of the total launch mass. And the engineering is fairly easy.

If the structure is 7% of the mass, the payload goes up to 2.3% of launch mass.

Two-Stage Solid Rocket

The velocity goal is 4500 m/s per stage with V_X = 2500 m/s.

That gives $\dfrac{m_1}{m_2} = e^{\frac{v_2}{V_X}} = e^{\frac{4500}{2500}} = 6.05.$

So the non-fuel mass can be 1 / 6.05 = 16.5% of the launch mass.

The Shuttle solids had a non-fuel mass of 15%,
so the second stage mass could only be 1.5% of the launch mass.

And the payload could only be 1.5% of that, or 0.02% of the launch mass.

3 & 4 Stage Solid Rocket

The velocity goal is 3000 m/s per stage with $V_X = 2500$ m/s.

That gives $\dfrac{m_1}{m_2} = e^{\frac{v_2}{V_X}} = e^{\frac{3000}{2500}} = 3.32$.

So the non-fuel mass can be 1 / 3.32 = 30.1% of the launch mass.

With non-fuel mass of 15%, the second stage could be 15% of the launch mass.

And the third stage mass could be 15% of that.
And the payload mass could be 15% of that.
The payload ends up as 0.34% of the launch mass.
Not very impressive, but military-surplus solid rockets are cheap…

A 4-stage solid rocket ends up at 0.8% payload mass.
There was a very popular all-solid 4-stage satellite launcher called Scout.

Extra Credit !

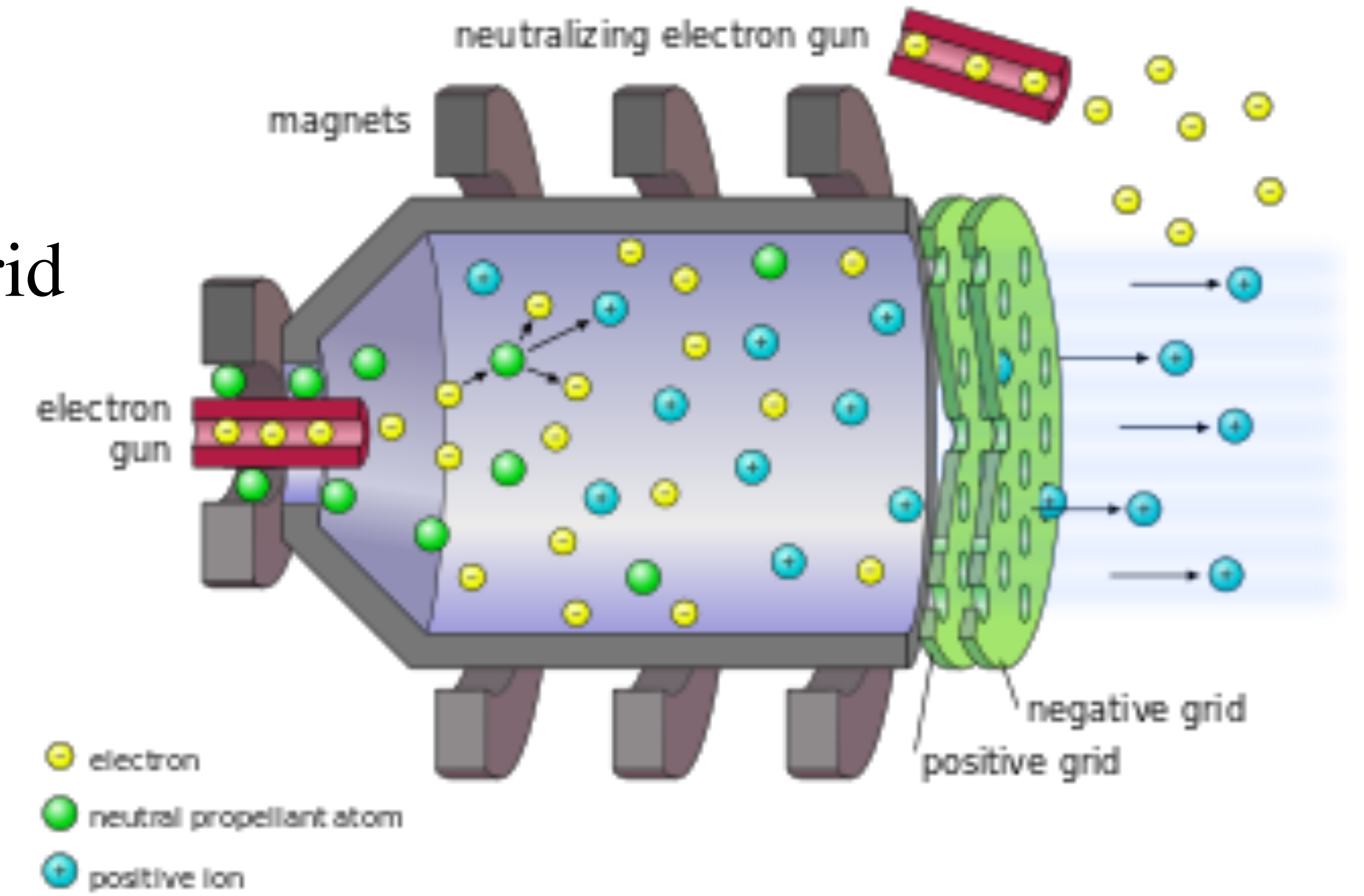
Electrostatic Rocket Engine

Ionize a gas, often Xenon

Attract the positive ions to a negative grid

Most ions exit the engine at high speed.

Spray electrons into the exhaust
(otherwise the ions get attracted back)



An ion with charge q accelerated by voltage U, gets kinetic energy qU.

For an ion with mass m, the exhaust velocity V_X can be calculated by

$$qU = \frac{1}{2} m V_X^2 \rightarrow V_X = \sqrt{\frac{2qU}{m}}$$

Electrostatic Rocket Engine 2

The equation $V_X = \sqrt{\dfrac{2qU}{m}}$ contains the electron charge and ion mass, atomic-physics things which are orders of magnitude away from SI units.

In atomic physics, the useful energy unit is the electron-Volt, eV, which is the energy that a particle with the charge of an electron gains when accelerated by a voltage difference of one Volt.

So in eV-units, the qU factor is just the charge q in electron units (1 or a few) times the voltage U in conventional Volts.

Using Einstein's $E = mc^2$, we get $m = E/c^2$.

If we use eV units for energies, the mass of a particle has dimensions eV/c^2.
The atomic mass unit (1/12 the mass of Carbon-12) is 931.5 MeV/c^2.

For an ion with atomic number A, $mc^2 = A \cdot 931.5\ \mathrm{MeV}$.

Electrostatic Rocket Engine 3

Put a factors of c^2 into $V_X = \sqrt{\dfrac{2qU}{m}}$ which gives $V_X = \sqrt{\dfrac{2qU \cdot c^2}{m \cdot c^2}}$.

Pull the numerator c^2 out of the square root, which gives $V_X = c \cdot \sqrt{\dfrac{2qU}{mc^2}}$

Use eV units, and plug in mc^2 for an ion with atomic number A,

giving $V_X = c \cdot \sqrt{\dfrac{2 \cdot \left(qU\right)_{\text{eV}}}{A \cdot 931.5 \text{ MeV}}}$.

Put in the value of c, and split up the square root

$$V_X = 3 \times 10^8 \ \frac{\text{m}}{\text{s}} \cdot \sqrt{\frac{2}{931.5 \times 10^6 \ \text{eV}}} \cdot \sqrt{\frac{\left(qU\right)_{\text{eV}}}{A}} = 1.39 \times 10^4 \, \frac{\text{m}}{\text{s}} \cdot \sqrt{\frac{\left(qU\right)_{\text{eV}}}{A}}$$

For Xenon with $A = 54$ and charge $q = 1$, if $U = 54$ Volts, $V_X = 13.9$ km/s

Exhaust velocities of 20-50 km/s are achieved in maneuvering satellites.

Fusion Torchship !

Deuterium plus Helium-3 fuse to Helium-4 plus a proton plus $E = 18.3$ MeV. Both the proton and Helium can be directed by a magnetic nozzle.

Helium is ~ 4 times the proton mass, so by conservation of momentum, Helium gets ~ 1/4 the proton velocity (the initial velocities are negligible).

The energy release E, proton mass m_p, and proton velocity v_p are related by

$$E = \frac{1}{2} m_p v_p^2 + \frac{1}{2} \cdot 4 m_p \cdot \left(\frac{1}{4} v_p \right)^2 = \frac{1}{2} m_p v_p^2 \cdot \left(1 + \frac{1}{4} \right)$$

It's useful to write this as $E = \frac{5}{8} \cdot \left(m_p c^2 \right) \cdot \left(\frac{v_p}{c} \right)^2 \rightarrow v_p = c \cdot \sqrt{ \cdot \frac{8}{5} \cdot \frac{E}{m_p c^2}}$

The proton mass factor is $m_p c^2 = 938.3 \text{ MeV}$, so

$$v_p = c \cdot \sqrt{\frac{8}{5} \cdot \frac{18.3 \text{ MeV}}{938.3 \text{ MeV}}} = 17.7\% \cdot c = 5.300 \times 10^7 \text{ m/s} = 53{,}000 \text{ km/s } !!$$

Fusion Torchship ! 2

Things aren't quite that good, because 4/5 of the exhaust mass is Helium.

Each He nucleus has the same momentum as a proton, but 1/4 the velocity.

So the mass-weighted average exhaust velocity is

$$\frac{1}{5}\cdot 5.3\times 10^4 + \frac{4}{5}\cdot\frac{1}{4}\cdot 5.3\times 10^4 = \frac{2}{5}\cdot 5.3\times 10^4 = 21{,}000 \text{ km/s !!}$$

That's pretty impressive, compared to escape velocity of 11 km/s.

(Epstein Drives get 10,000 to 20,000 km/s, so they didn't cheat in The Expanse)

Unfortunately, fusion reactors aren't practical yet, especially for D-He-3.

$\mathrm{D} + \mathrm{T} \rightarrow {}^4\mathrm{He} + \mathrm{n}$ would give about half the exhaust velocity because the neutron can't be turned into thrust by magnets. It's also not practical, although closer. It's hard to shield from the neutrons, and they damage and activate materials.