Einstein's theory of special relativity

In the late 1800's, Maxwell successfully developed a unified theory of electromagnetism, however, his equations required that the speed of light be constant in all directions and be independent of the velocity of its source. Maxwell's result was the cause of confusion among physicists regarding the nature of light until Einstein published his paper on Special Relativity in 1905.

1. Imagine you are on a train which is travelling at a constant speed of v. You then throw a ball out of the train at a speed of u in the same direction. How fast do you see the ball moving? What is the speed of the ball from the perspective of a person standing still on the ground outside of the train? Use the following values of v and u:
   a. v=50m/s, u=10m/s
   b. v=0.8c, u=c (imagine the ball is a photon)

   Hint: don't worry if your answer doesn't make sense, just add the velocities like you normally would and we can fix any issues later in the problem set.

   a) 50m/s + 10m/s = 60m/s

   b) c + 0.8c = 1.8c

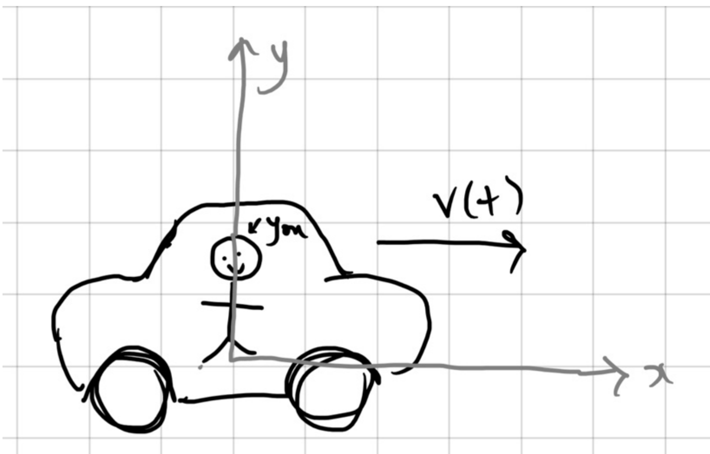
Part b of that question poses an important problem for our understanding of light and relativity; we know that photons can't travel faster than c! We can look to Einstein to help resolve (some) of the confusion.

Einstein's first two postulates of special relativity:

The laws of physics are the same in all inertial reference frames.
The speed of light is a constant, independent of the speed of its source.

We will use these two statements as the starting point for the rest of our calculations, but what do they tell us about physics?

Firstly, it is important to understand what a reference frame is. Imagine that you're in a car, driving along at some speed. You want to describe the world as you see it from the car. For simplicity, you choose the origin of your coordinate system to be where you are so that when you move, the origin of your coordinate system moves with you.



The things that are with you in the car will probably have a constant position. You can look out the window and write expressions for the position of all the objects you see in your coordinate system. In your coordinate system, they will appear to be moving backwards, even though you are technically the one moving. This is a super important concept in special relativity! Any object can have its own coordinate system (or **reference frame**) which describes the positions and physics of different things within it.

An **inertial reference frame** is, most simply, a reference frame (coordinate system) that is not accelerating. So, a car that is driving in a straight line with a constant speed is an inertial reference frame, but a car that is turning a corner or speeding up or slowing down is not. The **rest frame** of an object is the reference frame in which that object is at rest (not moving with respect to the observer).

Now what does it mean to say that the laws of physics are the same in all inertial reference frames? Imagine you are a in car, moving at a constant velocity, with all the windows in the car covered up. Can you tell if you are moving?

The correct (but perhaps unintuitive) answer is no!

Now imagine that you are in a car which is speeding up, slowing down, or turning a corner. Can you tell if you are moving in this situation?

What makes these situations different? In the first, you are in an inertial reference frame. In the second, you are not.

But what in particular gives you the intuition about when you're in a non-inertial reference frame? When a car speeds up, slows down, or turns a corner, you are usually pushed forward, back, or to the side, it feels like there is some imaginary force that's pushing you. In an elevator accelerating upwards, it might feel like gravity has gotten stronger (or weaker if you're going downwards).

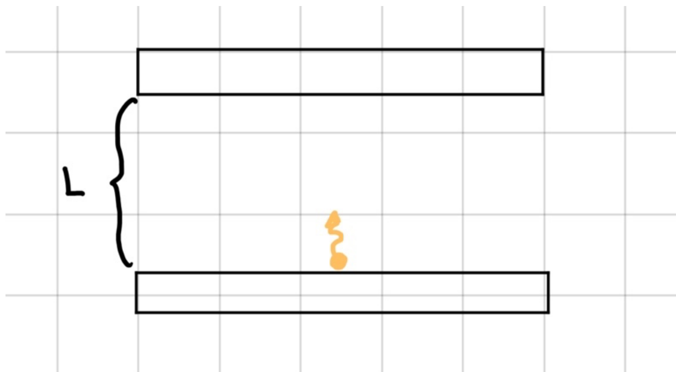
This is what Einstein is talking about when he says the laws of physics are the same in all inertial reference frames. To do physics in non-inertial reference frames we have to add what are called **fictitious forces**, to our normal laws (gravitation, electromagnetism, etc.) of physics.

You might be familiar with this concept from circular motion. Maybe you have heard your physics teacher make a distinction between centripetal and centrifugal force. Centripetal force is not a fictitious force, it is just the sum of all the different forces acting on a spinning object (form gravity, the thing spinning it, etc.). But centrifugal force is a fictitious force. It's what you would feel in a car going around a corner due to your inertial resistance to changing velocity. Inertia is what is causing this feeling, but inertia is not a force.

Now that we've defined the important terms, Einstein's second postulate is pretty self-explanatory: the speed of light is a constant (c in a vacuum) in all inertial reference frames.

Let's move on to some problems where we can apply these postulates.

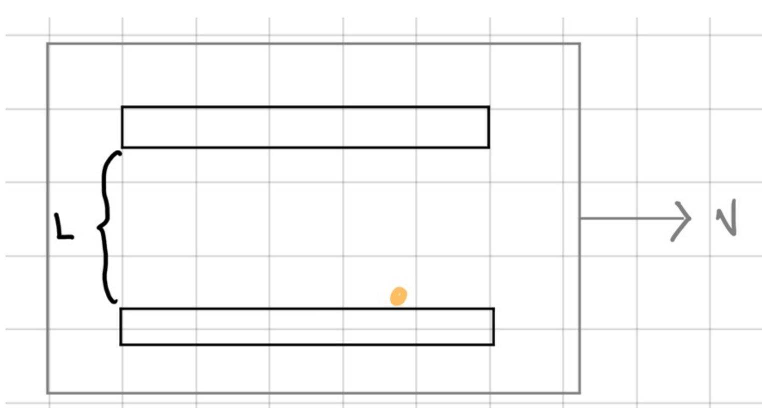
2. Consider a device with two parallel mirrors separated by a distance L.
   a. An observer in the rest frame of the device wants to measure how long it takes a photon to reflect off the top mirror and return to its starting point at the bottom mirror. How long does this observer measure the light to travel for?



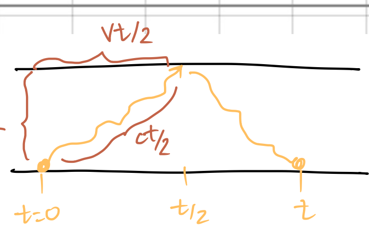
distance travelled $= 2L$, speed $= c$

$$t = \frac{2L}{c}$$

   b. An observer outside of the reference frame of the device sees it moving to the right at a speed of v. They can see the light from outside and want to measure the same thing. How long does this observer measure the light to travel for? Your answer should be in terms of L, c, and v.
   Hint: first sketch the path that the observer sees the light take



path the light takes :



Using Pythagoras, we get:

$$\left(\frac{vt}{2}\right)^2 + L^2 = \left(\frac{ct}{2}\right)^2$$

$$L^2 = \left(\frac{ct}{2}\right)^2 - \left(\frac{vt}{2}\right)^2$$

$$L^2 = \frac{t^2}{4}\left(c^2 - v^2\right)$$

$$t^2 = \frac{4L^2}{c^2 - v^2}$$

$$t = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L}{c}\frac{1}{\sqrt{1 - v^2/c^2}}$$

c. Let's name the reference frame from inside the device (from part a) A and the reference frame outside the device (from part b) B. What is the ratio of the light travel time $\frac{t_B}{t_A}$?

This ratio should be a function of only v and c

$$t_B = \frac{2L}{c}\frac{1}{\sqrt{1 - v^2/c^2}} \qquad t_A = \frac{2L}{c}$$

so, $$\frac{t_B}{t_A} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma$$

This ratio is called the gamma factor, γ, and is very important for relativistic calculations. It tells us that the time *we see* pass in an inertial reference frame which is moving with respect to us is *greater* than the time that passes from their perspective in their own reference frame.

Let t' represent the time that we see pass in a moving reference frame and let t_p represent the time that an observer *inside* the moving reference frames sees pass (t_p is often called the **proper time** since it is the time measured in the mirror's rest frame), we can relate these quantities by the **time dilation** formula:

$$t' = \gamma t$$

3. What is γ for $v = 0$? What about for $v = c$? What does this tell you about the importance of considering relativistic effects in different situations?

$\gamma \to 1$ as $v \to 0$

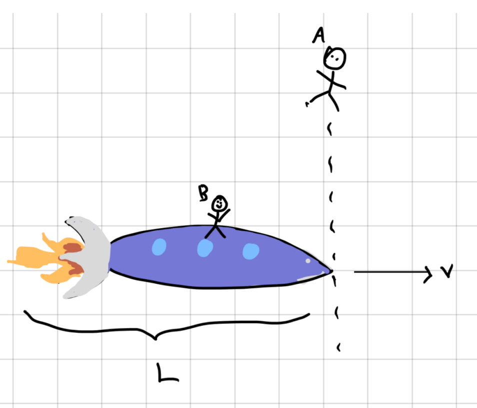
$\gamma \to \infty$ as $v \to c$

relativistic effects are non-negligible when v is a significant fraction of c

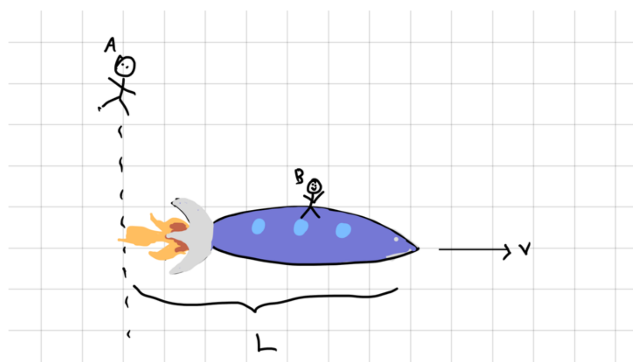
Now let's look at the effects of Einstein's postulates on the length of objects in difference reference frames.

4. Consider Bob who is on a spaceship travelling through the universe at a speed v. They come across Alice, who is floating around in space. Let the moment that the front of their spaceship lines up Alice be $t = 0$. In Bob's reference frame, the length of the spaceship is L. This is the proper length of the asteroid since it is the length perceived in its rest frame.

t=0s



t=??s



a. At what time t does Bob see the back of the spaceship pass Alice?

Bob sees the length of the spaceship to be L, so

$$L_{BOB} = v t_{BOB}$$

$$\Rightarrow \quad v t_{BOB} = L$$

b. At what time t does Alice see the back of the spaceship pass them? How long does Alice think the spaceship is?

We know $t' = \gamma t$

now we switch to Alice's frame so that

$t' = t_{BOB}$ and $t = t_{Alice}$ (Alice's time is the proper time)

$$t_{BOB} = \gamma t_{ALICE} \Longrightarrow t_{Alice} = \frac{t_{BOB}}{\gamma} = \frac{L}{\gamma v}$$

so Alice thinks the ship's length is: $L_{Alice} = t_{Alice} \cdot v = \frac{L}{\gamma}$

c. What is the ratio of lengths seen by Alice and Bob, $\frac{l_A}{l_B}$?

$$L_A = \frac{L}{\gamma} \quad L_B = L \Rightarrow \frac{L_A}{L_B} = \frac{1}{\gamma}$$

d. Write the equation for **length contraction** by expressing L' (the length of an object seen by a moving observer) in terms of L (the length of an object in its rest frame).

$$L' = \frac{1}{\gamma} L$$

Ask one of us to double check your equations for time dilation and length contraction before you continue.

5. Two twins, Alice and Bob, are 10 years old when Alice leaves the Earth to go on an interstellar trip. Alice travels to a star which is 10lyr away from Earth in a spaceship that travels at 0.8c.

   a. In Bob's frame, how old are each of the twins when Alice reaches the star?

   In Bob's frame, the star is 10lyr away and Alice travels at 0.8c

   $t_{BOB} = 10\text{lyr} \cdot 0.8c = 12.5\text{yrs}$

   Bob sees $t_{Alice} = \gamma t_{BOB}$ since he think Alice is moving towards the star

   $t_{Alice} = \gamma(0.8c)(12.5\text{yr})$

   $= \frac{12.5}{\sqrt{1-0.8^2}}$

   $= 12.5\left(\frac{5}{3}\right)\text{yrs}$

   $= 20.83\text{yrs}$

   so Bob thinks he is 22.5 years old and Alice is 30.8 years old.

   b. In Alice's frame, how old are each of the twins when Alice reaches the star?

   In Alice's frame, the star is moving towards her and Bob is moving away from her

   So the distance to the star is length contracted

   $D' = \frac{1}{\gamma} D \Rightarrow D' = \frac{10}{5/3} = 6\text{lyr}$

   so Alice sees pass for herself is:

   $t_{Alice} = 6\text{lyr} \cdot 0.8c = 7.5$ years pass for herself and the time that Alice sees pass for Bob is:

$$t_{BOB} = \gamma t_{Alice} = \frac{5}{3}(7.5) = 12.5 \text{ years}$$

So Alice thinks she is 17.5 years old and Bob is 22.5 years old

c. Once Alice reaches the star, she comes back to Earth. How old are each of twins when Alice gets back to Earth? This is called the Twin Paradox. Can you resolve it?

In Bob's frame
Bob is $10 + (12.5 \times 2) = 35$
Alice is $10 + (20.83 \times 2) = 52$

In Alice's frame
Bob is $10 + (12.5 \times 2) = 35$
Alice is $10 + (7.5 \times 2) = 25$

Alice thinks Bob has aged more than her, but Bob thinks Alice has aged more ??

Einstein's postulates say that the speed of light is constant in all inertial reference frames

But Alice's frame isn't inertial!

When she reaches the star, she has to turn around, which means she must be accelerating for some part of her trip.

We would have to use general relativity to do this calculation