Physics Circle Dec 4, 2024 Problem Set

Exploring the 2D Ising Model

Introduction

The Ising model is a mathematical framework used to study magnetism. Spins represent small magnetic moments of particles that can point up (+1) or down (−1). These spins interact with their neighbors and external magnetic fields. The alignment of spins gives rise to the magnetism observed in typical everyday magnets.

The energy of a system in the Ising model is given by:

$$E = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

where:

- J is the interaction strength between neighboring spins ($J > 0$ promotes alignment),
- h is the external magnetic field strength,
- $S_i = \pm 1$ represents the spin of the i-th atom,
- $\sum_{\langle i,j \rangle}$ indicates the sum over all neighboring spin pairs.

Questions

Part 1: Energy Calculation for a Small Grid

Consider a 2×2 grid of spins, where all spins are initially pointing up (+1). Calculate the total energy of the system for the following cases:

1. $J = 1$, $h = 0$
2. $J = 1$, $h = 2$

Show all steps clearly.

Part 2: Role of Temperature in Spin Alignment

At high temperatures, spins are random, while at low temperatures, they align. Using the concept of energy minimization, explain why the system prefers all spins to align at low temperatures when $J > 0$.

Part 3: Critical Temperature and Everyday Magnets

The 2D Ising model predicts a critical temperature (T_c) at which a material transitions from being magnetized to unmagnetized. Research and explain how this concept relates to typical everyday magnets. Hint: Think about how heating a magnet affects its properties.

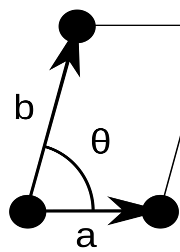
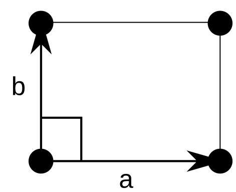
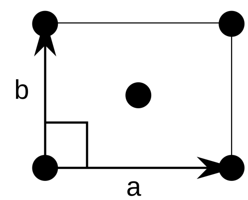
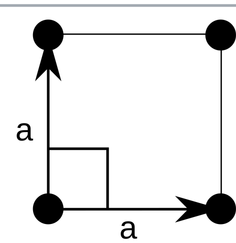
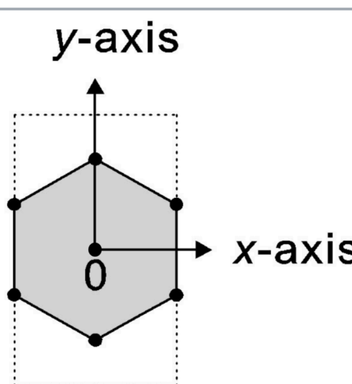
Part 4: External Field Influence

Now consider the same 2×2 grid, but with one spin flipped to -1. Assume $h = 2$. Calculate the new total energy of the system. Based on your calculation, explain how an external magnetic field influences spin alignment.

Part 5 (Bonus): Application to Modern Physics

The Ising model has applications beyond magnetism, such as neural networks, economics, and biology. Speculate on how the concept of interacting elements (spins) could be used to model other systems in physics or beyond.

Table referenced from Wikipedia page “Bravais Lattice”

Lattice system	**5 Bravais lattices**	
	Primitive (p)	**Centered (c)**
Monoclinic (m)	 Oblique	None
Orthorhombic (o)	 Rectangular	 Centered rectangular
Tetragonal (t)	 Square	None
Hexagonal (h)		None

Question 2: Bravais Lattice in 2D

Bravais lattice is a topic in geometry and study of crystals. Bravais lattice is an array of discrete points which repeat to form crystal structure. All crystal arrangements can be expressed by these lattices where we have atoms in all the lattice points of the Bravais lattice. More complicated crystals structures can be expressed as Bravais Lattice with a basis but we will not discuss them today.

All lattice points of Bravais lattice can be expressed by a lattice vector:

$$R = n_1\vec{a}_1 + n_2\vec{a}_2$$

Where n_1 and n_2 and integers (both positive or negative) and $\vec{a}_1\ and\ \vec{a}_2$ are vectors in the 2D plane. In the table in the previous page, the lattice $\vec{a}_1\ and\ \vec{a}_2$ are drawn in the lattice itself, represented by two arrows of length a or b.

For example, in Tetragonal or square lattice, the two lattice vectors are: $\vec{a} = a\hat{x}$ & $\vec{a}_2 = a\hat{y}$.

Part (a)

Notice that the lattice vectors are not drawn in the picture given for the hexagonal lattice. Can you write down 2 lattice vectors for this lattice?

Note that your lattice vector $\vec{a}_1\ and\ \vec{a}_2$ is only valid if all lattice points are accessible from each other, for some integers n_1 & n_2 in $R = n_1\vec{a}_1 + n_2\vec{a}_2$. (i.e. all the lattice points must be separated by vector $\vec{R}$.)

Part (b)

Lattice vectors are not unique and often can be written differently. In the example of the hexagonal lattice, can you find two alternative lattice vectors $\vec{a}_1$ and $\vec{a}_2$?

Part (c):

Notice that under orthorhombic lattices in the table (row 2), there is a rectangular lattice and a centered rectangular lattice (In the centered rectangular, there is an additional lattice point in the center of the rectangle.) However under tetragonal lattices (row 3), there only exists a square lattice and no centered square lattice.

Can you explain why a centered square lattice is NOT a separate Bravais Lattice?
(**Hint**: draw out many connected square centered lattices. Now see if you can describe this repeating pattern that you have drawn in terms of any of the other Bravais Lattices in the table.)