Solution

Part 1: Energy Calculation for a Small Grid

The energy formula is:

$$E = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

Case 1: $J = 1$**,** $h = 0$

- Spins are all $+1$. - There are 4 nearest-neighbor pairs in the 2×2 grid. Each spin interacts with its neighbors. - For each pair $S_i S_j = 1$, and there are 4 pairs:

$$\sum_{\langle i,j \rangle} S_i S_j = 4$$

Substitute into the energy formula:

$$E = -J \cdot 4 - h \cdot \sum_i S_i = -1 \cdot 4 - 0 \cdot 4 = -4$$

Answer: $E = -4$

Case 2: $J = 1$**,** $h = 2$

- Spins are still all $+1$. - Neighbor interaction is the same as before: $\sum_{\langle i,j \rangle} S_i S_j = 4$. - External magnetic field contribution:

$$\sum_i S_i = 4$$

Substitute into the energy formula:

$$E = -J \cdot 4 - h \cdot \sum_i S_i = -1 \cdot 4 - 2 \cdot 4 = -4 - 8 = -12$$

Answer: $E = -12$

Part 2: Role of Temperature in Spin Alignment

At high temperatures, thermal energy causes spins to randomize, leading to no net magnetization. At low temperatures, the system minimizes energy by aligning spins (ferromagnetic order) when $J > 0$.

Part 3: Critical Temperature and Real-World Magnetism

The critical temperature (T_c) corresponds to the Curie temperature in ferromagnetic materials like iron. Above T_c, thermal energy disrupts spin alignment, destroying magnetization. Below T_c, spins align spontaneously, creating a net magnetic moment.

Part 4: External Field Influence

- With one spin flipped (-1) and $h = 2$:

$$\sum_{\langle i,j \rangle} S_i S_j = 2, \quad \sum_i S_i = 2$$

Substitute into the energy formula:

$$E = -J \cdot 2 - h \cdot 2 = -1 \cdot 2 - 2 \cdot 2 = -6$$

Answer: $E = -6$

Part 5: Application to Modern Physics

The Ising model applies to neural networks (spins = neurons), economics (spins = agents), and biology (e.g., protein folding). It demonstrates how local interactions create complex systems, aiding understanding across disciplines.

1 Question 2 Solutions

Part(a) While the lattice vectors are non unique, we can choose a lattice vector as such:

$$\vec{a_1} = a\hat{x}$$

$$\vec{a_2} = \frac{a}{2}\hat{x} + \frac{\sqrt{3}}{2}a\hat{y}$$

Part (b) Notice that if we rotate our page 90 degrees, then the hexagonal lattice is unchanged. We can use this symmetry to write another equivalent lattice vector:

$$\vec{a_1} = a\hat{y}$$

$$\vec{a_2} = \frac{\sqrt{3}}{2}a\hat{x} + \frac{a}{2}\hat{y}$$

Part (c)

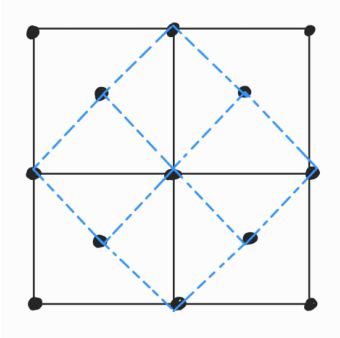

Figure 1: square centered lattice

A centered square lattice will have an additional point in the center compared to the square lattice. As shown in the diagram above, the black solid lines are how we would normally draw the square lattice if there was no lattice point at the center. However, given there is a lattice point in the center, we can redraw the lattice as shown by the blue dashed lines instead. For the blue dashed lines, all the angles are still 90 degrees, but the lattice size is slightly smaller. However this means that the centered square lattice can ve more simply represented as a square lattice with smaller lattice area. Thus, it is not a new type of Bravais lattice.